Partial Logics with Two Kinds of Negation as a Foundation for Knowledge-Based Reasoning

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1 Introduction

As opposed to theoretical reasoning, such as in mathematics, where all predicates are *exact*,[1] and a single contradiction destroys the entire theory, *knowledge-based reasoning* has to be able to deal with inexact predicates (e.g. from empirical domains) having truth value gaps, and with knowledge bases containing contradictory items but being still informative. Therefore, partial logics allowing both for truth-value gaps and for inconsistency are natural candidates for modelling knowledge-based reasoning.

In knowledge representation, two different notions of falsity arise in a natural way. Certain facts are *implicitly false by default* by being not verified in any intended model of the knowledge base. Others are *explicitly false* by virtue of a direct proof of their falsity, corresponding to their falsification in all intended models. These two kinds of falsity in knowledge representation are captured by the two negations, called *weak* and *strong*, of partial logic.[2] In the monotonic base system of partial logic, weak negation corresponds to classical negation by virtue of a straightforward translation of partial logic into classical logic which is discussed in section 3. In the nonmonotonic refinements of partial logic, discussed in sections 4 and 5, weak negation corresponds to negation-as-failure, and hence can be used to express local Closed-World Assumptions, default rules, and the like.

As opposed to the traditional logical notion of a *theory* being a (possibly deductively closed) set of formulas, the emerging concept of a *knowledge base (KB)* is richer both in terms of the expressive structure of a KB and in terms of the meaningful restrictions imposed upon it. Typically, a KB consists of *facts* and various kinds of *rules*. In this paper, we shall only consider *deduction rules*. Facts correspond to sentences of an appropriately restricted language, and deduction rules correspond to non-schematic (Gentzen) sequents. While facts express extensional knowledge, rules express intensional knowledge. This dichotomy of the knowledge representation language also affects the use of the universal quantifier: a generic law, for instance, is rather expressed in the form of a rule and not by means of a universal sentence.

In real world knowledge bases like, for instance, relational or deductive databases, it is essential to be able to infer negative information by means of *minimal* (resp. *stable*) reasoning, i.e. drawing inferences on the basis of minimal (resp. stable) models. Relational databases, being finite sets of tables the rows of which represent atomic sentences, have traditionally been viewed as finite models. On this account, answering a query F is rather based on the model relation, $\mathcal{M}_\Delta \models F$, where $\mathcal{M}_\Delta$ is the finite interpretation corresponding to the database Δ, and not on an inference relation. However, especially with respect to the generalization of relational databases (e.g. in order to allow for incomplete information), it seems to be more adequate to regard a relational database as a set of atomic sentences A_Δ, and to infer a query F whenever it holds in the unique minimal model of

[1] In the sense of Körner [Koe66].
[2] This was already noticed in [Wag91].

A_Δ, i.e.

$$A_\Delta \vdash F :\Leftrightarrow \mathrm{Min}(\mathrm{Mod}(A_\Delta)) \subseteq \mathrm{Mod}(F) \Leftrightarrow \mathcal{M}_\Delta \models F$$

While minimal models are adequate for definite extensional knowledge bases (such as relational databases), a refinement of the notion of minimality, called *paraminimality*, is needed to capture the inclusiveness of disjunctive knowledge. Minimal and paraminimal models are discussed in section 4.

It turns out, that for a deductive knowledge base, corresponding to a set of sequents, minimal (resp. paraminimal) models are not adequate because they are not able to capture the directedness of rules. We, therefore, propose *stable* models as the intended models of deductive knowledge bases in section 5. We show that Gelfond's and Lifschitz's notion of an *answer set* of an extended logic program [GL90] corresponds to a special case of our notion of a stable model of a sequent set.

Since in practice large knowledge bases cannot be expected to be free of inconsistent information, one needs a notion of inference which is able to tolerate inconsistency and at the same time still as logically conservative as possible. In order to deal with possibly inconsistent KBs, the simplest way is to refer to *minimally inconsistent* four-valued models as proposed in [Pri89]. In summary, we get an 'orthogonal' combination of minimally inconsistent paraminimally stable models as the preferred models of a deductive knowledge base.

2 Preliminaries

A *signature* $\sigma = \langle Rel, ExRel, Const, Fun\rangle$ consists of a set of relation symbols *Rel*, a set $ExRel \subseteq Rel$ of exact relation symbols, a set of constant symbols, and a set of function symbols.

The set of all variables, *Var*, is $\{x_0, x_1, \ldots\}$; we will also use $x, y, \ldots$, however. $U(\sigma)$ denotes the set of all ground terms of σ. The logical functors are $-, \sim, \wedge, \vee, |, \supset, \forall, \exists$; where $-, \sim, |$ and $\supset$ are called *weak* negation, *strong* negation, *exclusive* disjunction, and *material implication*, respectively.[3] $L(\sigma)$ is the smallest set containing the atomic formulas of σ, and being closed with respect to the following conditions: if $F, G \in L(\sigma)$, then $\{\sim F, -F, F \wedge G, F \vee G, F|G, F \supset G, \exists x F, \forall x F\} \subseteq L(\sigma)$.

$L_0(\sigma)$ denotes the corresponding set of sentences (closed formulas). For sublanguages of $L(\sigma)$ formed by means of a subset $\mathcal{F}$ of the logical functors, we write $L(\sigma; \mathcal{F})$. With respect to a signature σ we define the following sublanguages: $\mathrm{At}(\sigma) = L(\sigma; \emptyset)$, the set of all atomic formulas (also called *atoms*); $\mathrm{Lit}(\sigma) = L(\sigma; \{\sim\})$, the set of all *literals*; $\mathrm{Lit}_0(\sigma)$ the set of ground literals (also called *Herbrand basis*), and $\mathrm{XLit}(\sigma) = \mathrm{Lit}(\sigma) \cup \{-l : l \in \mathrm{Lit}(\sigma)\}$, the set of all *extended literals*. We introduce the following conventions. When $L' \subseteq L(\sigma)$ is some sublanguage, L'_0 denotes the corresponding set of sentences. If the signature

[3] Possible extensions of our framework may in addition include negation-as-inconsistency ($\neg$), intensional implication ($\rightarrow$), and modal operators for definite and persistent belief.

σ does not matter, we omit it and write, e.g., L instead of $L(\sigma)$. Furthermore, $\widetilde{X} = \{\sim F : F \in X\}$.

Let $L \subseteq L(\sigma)$ be a nonempty language. An operation $C : 2^L \to 2^L$ is called an *inference operation*, and the pair $\langle L, C\rangle$ is said to be an *inference system*. The corresponding *inference relation* $\vdash$ is defined by $X \vdash F$ iff $F \in C(X)$. An inference operation (relation) is called a *consequence* operation (relation) if it satisfies Inclusion (Reflexivity), Idempotence (Transitivity), and Monotony. $\langle L, C\rangle$ is called a *deductive system* if C is a consequence operation satisfying Compactness.

A *model-theoretic system* $\langle L, \boldsymbol{I}, \models\rangle$ is determined by a language L, a set $\boldsymbol{I}$ whose elements are called *interpretations* and a *model relation* $\models \subseteq \boldsymbol{I} \times L$ between interpretations and formulas. With every model-theoretic system $\langle L, \boldsymbol{I}, \models\rangle$, we can associate a model operator Mod_I, a consequence operation C_I, and a consequence relation $\models_I$ in the following way. Let $X \subseteq L$, then the associated model operator is defined as $\mathrm{Mod}_I(X) = \{\mathcal{I} \in \boldsymbol{I} : \mathcal{I} \models X\}$, where $\mathcal{I} \models X$ iff for every $F \in X : \mathcal{I} \models F$. The associated consequence operation is defined by $C_I(X) = \{F \in L : \mathrm{Mod}_I(X) \subseteq \mathrm{Mod}_I(F)\}$, and finally $X \models_I F$ iff $F \in C_I(X)$. For a subset $\boldsymbol{K} \subseteq \boldsymbol{I}$ the *theory* of $\boldsymbol{K}$, denoted by $Th(\boldsymbol{K})$ is defined by $Th(\boldsymbol{K}) = \{F \in L : \mathcal{I} \models F \text{ f.a. } \mathcal{I} \in \boldsymbol{K}\}$. A model-theoretic system $\langle L, \boldsymbol{I}, \models\rangle$ is called *compact* if C_I is compact. An inference system $\langle L, C_L\rangle$ is called *correct*, resp. *complete*, with respect to the model-theoretic system $\langle L, \boldsymbol{I}, \models\rangle$ iff $C_L(X) \subseteq C_I(X)$, resp. $C_L(X) = C_I(X)$. In the case of completeness we also say that $\langle L, \boldsymbol{I}, \models\rangle$ *represents* $\langle L, C_L\rangle$.

If X is a set of sets, then $\mathrm{Fin}(X)$ denotes its restriction to finite elements. If Y is an partially ordered set, then $\mathrm{Min}(Y)$ denotes the set of all minimal elements of Y, i.e. $\mathrm{Min}(Y) = \{X \in Y \mid \neg\exists X' \in Y : X' < X\}$, and $\mathrm{Max}(Y)$ denotes the set of all maximal elements of Y, i.e. $\mathrm{Max}(Y) = \{X \in Y \mid \neg\exists X' \in Y : X' > X\}$.

3 Partial Logics with Two Kinds of Negation

In this section we start with a brief introduction of partial model-theory, and then we present their underlying axiomatics. Since partial logic adopts its name from its alternative at the very core of denotational semantics, consisting of a shift from total to partial truth-value assignments, this order of presentation seems most natural.

More specifically, we begin with a presentation of partial first-order models. Then we will discuss some issues of the expressivity of certain languages for reasoning on the basis of partial models. An essential feature of partial models is the fact that they allow to distinguish between two types of extensional[4] negative information, i.e. between the *explicit falsity* and the *non-truth* of a proposition.

[4]Roughly speaking, extensionality says that the information is only about one specific information state or model. Intensional information comes from other information state which are related in one way or another to the information state at hand. An example of an intensional treatment of negation can be found in intuitionistic logic. In this setting, $\neg\phi$ means that every hypothetical verification of ϕ will lead to a contradiction. In other words, for determining the

After this, we will show how partial first-order logics can be translated into classical first-order logic. This result does not mean that partial logic is abundant[5] but rather shows how well-known meta-theoretic theorems can be adopted from classical logic. An immediate consequence, which is directly relevant for this paper, is compactness.

In the third subsection we will present Gentzen-style axiomatizations of partial logics. Other styles of derivation, like Hilbert-style axiomatization and natural deduction, are also possible. The reasons for us to chose in favor of the Gentzen-style comes down to its meta-theoretical convenience and its brevity.

3.1 Model Theory

The model-theory of partial logic is slightly deviant from the standard Tarskian one of classical logic. The only difference is that the predicate structure is somewhat richer. As already stressed above, the central idea of partial logic is the distinction between falsity and non-truth. In the partial predicate logics which we will discuss this distinction is implemented by assigning a *positive* and a *negative* extension to each predicate.

Definition 1 (Interpretation) *Let $\sigma = (Rel, ExRel, Const, Fun)$ be a signature. A partial σ-interpretation $\mathcal{I}$ consists of:*

1. *A set $U_\mathcal{I}$, the* universe *or* domain *of $\mathcal{I}$;*
2. *an assignment $c^\mathcal{I} \in U_\mathcal{I}$ to every constant symbol $c \in Const$;*
3. *an assignment of a function $f^\mathcal{I} : U_\mathcal{I}^{ar(f)} \to U_\mathcal{I}$ to every function symbol $f \in Fun$, where $ar(f)$ denotes the arity of f;*
4. *an assignment of a pair $\langle R^\mathcal{I}, \widetilde{R}^\mathcal{I} \rangle$ to every relation symbol $R \in Rel$ such that*
$$R^\mathcal{I} \cup \widetilde{R}^\mathcal{I} \subseteq U_\mathcal{I}^{ar(R)},$$
and in the special case of an exact relation symbol $R \in ExRel$,
$$R^\mathcal{I} \cup \widetilde{R}^\mathcal{I} = U_\mathcal{I}^{ar(R)},$$
where $ar(R)$ denotes the arity of R.

While many predicates from the ontology of empirical domains are inexact, i.e. have truth value gaps, analytical predicates (such as equality, or being a prime

truth of $\neg\phi$ we need to take 'later' states of information, which contain more information than the current one, into account.

[5]Opponents of partial logic may argue that the translation actually 'proves' the abundance of partial logic. We disregard such an abstract position, because for practical purposes, partial logic arises as the most natural model-theoretic method for interpreting the two kinds of extensional negative information that we mentioned above.

number), and legally defined predicates (such as being eligible, or having a certain nationality) are exact.

In the sequel we shall often simply say 'interpretation' instead of 'partial interpretation'.

The class of all partial σ-interpretations is denoted by $\boldsymbol{I}_4(\sigma)$. We define the classes of *coherent* (sometime also called *3-valued*), of *total*, and of total coherent (or *2-valued*) σ-interpretations by

$$\begin{aligned}
&\boldsymbol{I}_c(\sigma) = \{\mathcal{I} \in \boldsymbol{I}_4(\sigma) : R^{\mathcal{I}} \cap \widetilde{R}^{\mathcal{I}} = \emptyset \text{ for all } R \in Rel\} \\
&\boldsymbol{I}_t(\sigma) = \{\mathcal{I} \in \boldsymbol{I}_4(\sigma) : R^{\mathcal{I}} \cup \widetilde{R}^{\mathcal{I}} = U_{\mathcal{I}}^{ar(R)} \text{ for all } R \in Rel\} \\
&\boldsymbol{I}_2(\sigma) = \boldsymbol{I}_c(\sigma) \cap \boldsymbol{I}_t(\sigma)
\end{aligned}$$

The satisfaction relation $\models$ between an interpretation, a valuation and a formula is defined inductively on the complexity of formulas $F \in L(\sigma)$ and $\sim F \in L(\sigma)$. Such a dichotomous induction is needed, because verification and falsification are independent truth-value assignments in partial logic.[6] A *valuation* over an interpretation $\mathcal{I}$ is a function $\nu : Var \to U_{\mathcal{I}}$, which can naturally be extended to arbitrary terms by

$$\nu(f(t_1, \ldots, t_n)) = f^{\mathcal{I}}(\nu(t_1), \ldots, \nu(t_n))$$

Note that for a constant c, being a 0-ary function, we have $\nu(c) = c^{\mathcal{I}}$. For a tuple $t_1, \ldots, t_n$ we will also write $\vec{t}$ when its length is of no relevance. We write $\mu =_x \nu$, if two valuations μ, ν are equal except for the variable x: $\mu(y) = \nu(y)$ for all $y \in Var \setminus \{x\}$.

Definition 2 (Satisfaction Relation)

$$\begin{array}{rcl}
\mathcal{I}, \nu \models R(t_1, \ldots, t_n) & \text{iff} & \langle \nu(t_1), \ldots, \nu(t_n) \rangle \in R^{\mathcal{I}} \\
\mathcal{I}, \nu \models \sim R(t_1, \ldots, t_n) & \text{iff} & \langle \nu(t_1), \ldots, \nu(t_n) \rangle \in \widetilde{R}^{\mathcal{I}} \\
\mathcal{I}, \nu \models F \wedge G & \text{iff} & \mathcal{I}, \nu \models F \text{ and } \mathcal{I}, \nu \models G \\
\mathcal{I}, \nu \models F \vee G & \text{iff} & \mathcal{I}, \nu \models F \text{ or } \mathcal{I}, \nu \models G \\
\mathcal{I}, \nu \models -F & \text{iff} & \mathcal{I}, \nu \not\models F \\
\mathcal{I}, \nu \models \forall x F & \text{iff} & \mathcal{I}, \mu \models F \text{ for all } \mu =_x \nu \\
\mathcal{I}, \nu \models \exists x F & \text{iff} & \mathcal{I}, \mu \models F \text{ for certain } \mu =_x \nu
\end{array}$$

All other cases of formula composition are treated by the following DeMorgan-style rewrite rules expressing the falsification of compound formulas:

$$\begin{array}{rclcrcl}
\sim (F \wedge G) & \longrightarrow & \sim F \vee \sim G & \quad & \sim (F \vee G) & \longrightarrow & \sim F \wedge \sim G \\
\sim \exists x F(x) & \longrightarrow & \forall x \sim F(x) & & \sim \forall x F(x) & \longrightarrow & \exists x \sim F(x) \\
\sim\sim F & \longrightarrow & F & & \sim -F & \longrightarrow & F
\end{array}$$

[6]Most often these two relations are also written in a different fashion, e.g. $\models$ for verification and $\Dashv$ for falsification. Such a treatment is needed when the strong negation $\sim$ is not available. In this paper, we will not deal with strong negation free sublanguages.

and the definitions for exclusive disjunction,

$$F|G \longrightarrow (F \wedge -G) \vee (G \wedge -F)$$

and material implication,

$$F \supset G \longrightarrow -F \vee G$$

in the sense that for every rewrite rule $LHS \longrightarrow RHS$, we define

$$\mathcal{I}, \nu \models LHS \quad \text{iff} \quad \mathcal{I}, \nu \models RHS$$

Notice that conjunction and disjunction, resp. the universal and the existential quantifier, are interdefinable via the DeMorgan rules, and consequently, it is sufficient in definitions and proofs to treat the functors $-, \sim, \wedge, \forall$.

Definition 3 (Model Relation) *The* model relation *between an interpretation and a formula $F \in L(\sigma)$ is also denoted by $\models$; it is defined by*

$$\mathcal{I} \models F \textit{ iff } \mathcal{I}, \nu \models F \textit{ for every } \nu \in {U_{\mathcal{I}}}^{Var}$$

If $\mathcal{I} \models F$ for every $F \in X$ and $\mathcal{I} \in \mathcal{I}_*$, then $\mathcal{I}$ is said to be a $*$-model of X.

For $* = 4, c, t, 2$, Mod_* denotes the model operator associated with the system $\langle L(\sigma), \boldsymbol{I}_*, \models \rangle$, and $\models_*$ and C_* denote the corresponding consequence relation and operation, i.e. $X \models_* F$ iff $\mathrm{Mod}_*(X) \subseteq \mathrm{Mod}_*(F)$. A set X is $*$-satisfiable iff $\mathrm{Mod}_*(X) \neq \emptyset$.

Definition 4 (Satisfaction Set) *Let $\mathcal{I} \in \boldsymbol{I}_4(\sigma)$, and $X \subseteq L(\sigma)$. Then*

$$\mathrm{Sat}_{\mathcal{I}}(X) = \{\nu \in {U_{\mathcal{I}}}^{Var} : \mathcal{I}, \nu \models X\}$$

Definition 5 (Logical Equivalence) *Let $F, G \in L(\sigma)$. The formulas F and G are logically $*$-equivalent, symbolically $F \equiv_* G$, iff for all $\mathcal{I} \in \boldsymbol{I}_*(\sigma)$, $\mathrm{Sat}_{\mathcal{I}}(F) = \mathrm{Sat}_{\mathcal{I}}(G)$.*

Note that this definition of equivalence does not capture uniform substitutability. For example $p \wedge \sim p \equiv_c q \wedge \sim q$, but $\sim(p \wedge \sim p) \not\models_c \sim(q \wedge \sim q)$. In general, substitutability of F by G can be regained by requiring that $F \equiv_* G$ and $\sim G \equiv_* \sim F$.

It is not hard to show that the general case of $\boldsymbol{I}_4(\sigma)$ can be reduced to classical logic. Because the propositions F and $\sim F$ are completely independent, they can be understood as two different propositions in a two-valued setting. This can be made explicit by a dichotomous translation function, which has been given (in a slightly different way) by Gilmore[Gil74], but can also be found in Feferman[Fef84] or Langholm[Lan88]. !!

Definition 6 (Gilmore translation) *The Gilmore translation function* $\mathbf{g}$ *is a pair* $\langle \mathbf{t}, \mathbf{f} \rangle$ *with:*

$$\begin{array}{rclrcl} (R(\vec{t}))^{\mathbf{t}} & = & R_{\mathbf{t}}(\vec{t}) & (R(\vec{t}))^{\mathbf{f}} & = & R_{\mathbf{f}}(\vec{t}) \\ (\sim F)^{\mathbf{t}} & = & F^{\mathbf{f}} & (\sim F)^{\mathbf{f}} & = & F^{\mathbf{t}} \\ (F \wedge G)^{\mathbf{t}} & = & F^{\mathbf{t}} \wedge G^{\mathbf{t}} & (F \wedge G)^{\mathbf{f}} & = & F^{\mathbf{f}} \vee G^{\mathbf{f}} \\ (\forall x F)^{\mathbf{t}} & = & \forall x F^{\mathbf{t}} & (\forall x F)^{\mathbf{f}} & = & \exists x F^{\mathbf{f}} \\ (-F)^{\mathbf{t}} & = & \sim F^{\mathbf{t}} & (-F)^{\mathbf{f}} & = & F^{\mathbf{t}} \end{array}$$

where we have introduced the new relation symbols $R_{\mathbf{t}}$ *and* $R_{\mathbf{f}}$ *which are intended to capture the truth and the falsity extension of* R*.*

If $\sigma = \langle \mathit{Rel}, \mathit{ExRel}, \mathit{Cons}, \mathit{Func} \rangle$ is a signature, then we define $\sigma^{\mathbf{g}}$ to be the signature $\langle \mathit{Rel}^{\mathbf{g}}, \mathit{Rel}^{\mathbf{g}}, \mathit{Cons}, \mathit{Func} \rangle$ such that $\mathit{Rel}^{\mathbf{g}} = \mathit{ExRel} \cup \{R_{\mathbf{t}}, R_{\mathbf{f}} \mid R \in \mathit{Rel}\}$. Furthermore, if $\mathcal{I}$ is a σ-interpretation, we write $\mathcal{I}^{\mathbf{g}}$ for the $\sigma^{\mathbf{g}}$-interpretation such that $\mathcal{I}$ and $\mathcal{I}^{\mathbf{g}}$ coincide with respect to *Cons* and *Func*, and for $R \in \mathit{Rel}$: $(R_{\mathbf{t}})^{\mathcal{I}^{\mathbf{g}}} = R^{\mathcal{I}}$, and $(R_{\mathbf{f}})^{\mathcal{I}^{\mathbf{g}}} = \widetilde{R}^{\mathcal{I}}$. By a simple inductive argument it can be shown that

$$\mathcal{I}, \nu \models F \quad \text{iff} \quad \mathcal{I}^{\mathbf{g}}, \nu \models F^{\mathbf{t}} \text{ for all } \mathcal{I}\text{-valuations } \nu. \tag{1}$$

The translation is surjective, which implies that we even have the following more drastic equivalences:

Proposition 1 *If* $X \subseteq L(\sigma)$ *and* $F \in L(\sigma)$*, then*

$$\begin{array}{lcl} X \models_4 F & \Longleftrightarrow & X^{\mathbf{t}} \models_2 F^{\mathbf{t}}; \\ X \models_c F & \Longleftrightarrow & X^{\mathbf{t}}, Y \models_2 F^{\mathbf{t}} \text{ with } Y = \{\sim(G^{\mathbf{t}} \wedge G^{\mathbf{f}}) \mid G \in L(\sigma)\}; \\ X \models_t F & \Longleftrightarrow & X^{\mathbf{t}}, Z \models_2 F^{\mathbf{t}} \text{ with } Z = \{G^{\mathbf{t}} \vee G^{\mathbf{f}} \mid G \in L(\sigma)\}.^7 \end{array}$$

Corollary 2 (Löwenheim-Skolem) *Let* $* = 4, t$ *or* c*. If a formula* $F \in L(\sigma)$ *is* $*$*-satisfiable, then it also has a countable model, i.e. there exists* $\mathcal{I} \in \mathrm{Mod}_*(F)$ *such that* $U^{\mathcal{I}}$ *is countable.*

Corollary 3 *Let* $* = 4, t$ *or* c*.*
(1) Compactness: $X \subseteq L(\sigma)$ *is* $*$*-satisfiable iff every finite subset of* X *is* $*$*-satisfiable.*
(2) Finiteness: $X \models_* F$ *iff there is a finite set* $Y \subseteq X$ *such that* $Y \models_* F$*.*

3.2 Propositional Expressivity and Normal Forms

Let us suppose that we only deal with the sublanguage $\mathit{Prop}(\sigma) := L_0(\sigma; \wedge, \vee, \sim, -)$. A σ-interpretation $\mathcal{I}$ can then be understood as a partial truth-value assignment $\mathcal{V}_{\mathcal{I}} : \mathit{At}_0(\sigma) \to 2^{\{0,1\}}$. The simple reason to do so is that we wish to discuss

[7] The stronger versions with $G \in \mathrm{At}(\sigma)$ also hold.

the expressivity of connectives, rather than that of quantifiers. The corresponding partial truth-value assignment: $\mathcal{V}_{\mathcal{I}}(P)$ is the subset of $\{0,1\}$ such that

$$\begin{aligned} 1 \in \mathcal{V}_{\mathcal{I}}(P) \quad &\text{iff} \quad P \in D_{\mathcal{I}} \\ 0 \in \mathcal{V}_{\mathcal{I}}(P) \quad &\text{iff} \quad {\sim}P \in D_{\mathcal{I}} \end{aligned}$$

In other words, $\{0,1\}$ stands for over-valued, $\{0\}$ for falsity, $\{1\}$ for truth, and $\emptyset$ for under-valued. The set of all truth-values, $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$, will be called **four**. The subsets $\{\emptyset, \{0\}, \{1\}\}$, $\{\{0\}, \{1\}, \{0,1\}\}$ and $\{\{0\}, \{1\}\}$ will be denoted by **three**, **three'** and **two**, respectively.

Of course, this definition settles a 1–1 correspondence between partial interpretations and partial truth-value assignments. For this reason, we will drop the $\mathcal{I}$-index in the sequel of this subsection. For the full collection of partial truth-value assignments we write $\mathbf{V_4}$. $\mathbf{V_c}$, $\mathbf{V_t}$ and $\mathbf{V_2}$ refer to the obvious subclasses of partial truth-assignments.

The question arises, whether our propositional language that we work with, is expressive enough to describe the content of a partial truth-assignment $V \in \mathbf{V_4}$. In other words, can every (extensional) connective be defined in terms of the connectives of the language. This property is also called *expressive* or *functional* completeness of the language. In classical logic, we know that the language $L_0(\sigma; \sim, \wedge)$ is adequate for this purpose. In partial logic this is certainly not the case, by means of these two connectives we can not express that a proposition is not true: $-P$ can not be defined by means of P, $\sim$ and $\wedge$ alone.

These issues of expressivity are not of purely theoretical concern. For example, given a subclass of models which behaves computationally very well, then we want to know the exact language which describes such a class.[8] Furthermore, if we want to axiomatize an extension of the model class $\boldsymbol{I}_2$, then we need to know whether connectives are independent or can be defined in terms of others. We know for sure, that the former class requires explicit reference within such an axiom system. Last but not least, we also want to have a formal understanding what we really gain in expressivity, once we extend a model class. For example, the formula $-(P \vee {\sim}P)$ has no 2-models, but is c-satisfiable, which makes clear that $-$ really adds expressive power to the connectives $\sim$ and $\vee$.

In other words, given a class of models, we wish to know the underlying languages of both super- and subclasses.

Formally, we interpret an n-ary connective γ as a function $[\gamma]$ from n-tupels of truth-values to truth-values.

$$[\gamma] : \mathbf{val}^n \longrightarrow \mathbf{val}$$

with **val** being one of the earlier mentioned truth-value sets: $\{\{0\}, \{1\}\} \subseteq \mathbf{val} \subseteq \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. For example, the weak negation $-$ is interpreted as the

[8] E.g. Langholm's description of Horn clauses in partial logic [Lan91] in terms of transferring the classical semantic properties of such clauses to partial logic, and then define the language which has this properties over partial models.

function

$$[-](x) = \begin{cases} \{0\} & \text{if } 1 \in x \\ \{1\} & \text{otherwise.} \end{cases}$$

The question arises, whether this weak negation is sufficient as an addition to $\sim$ and $\wedge$ to obtain functional completeness for the classes $\mathbf{V_c}$, $\mathbf{V}_t$ and $\mathbf{V_4}$. The answer is: "nearly". We only need to add some additional nullary connectives $\mathbf{u}$ and $\mathbf{o}$, which obtain the following denotation: $[\mathbf{u}] = \emptyset$, and $[\mathbf{o}] = \{0, 1\}$.

The following table presents for all four classes the associated set of connectives which yields functional completeness.

$\mathbf{V_2}$	$\sim, \wedge$
$\mathbf{V_c}$	$\mathbf{u}, \sim, -, \wedge$
$\mathbf{V_t}$	$\mathbf{o}, \sim, -, \wedge$
$\mathbf{V_4}$	$\mathbf{u}, \mathbf{o}, \sim, -, \wedge$

In the field of partial logic many more expressivity results are known for well-defined subclasses of $\mathbf{V_c}$ and $\mathbf{V_4}$ (see [Bla86], [Lan88], [Ben84], [Mus95] and [Thi92]). An important result is the functional completeness of $\mathbf{u}, \sim, \wedge$ with respect to the persistent connectives over $\mathbf{V_c}$ by Blamey in [Bla86].[9] A connective γ is persistent iff its interpretation $[\gamma]$ is monotone over $\subseteq$:

$$(\forall i \in \{1, \ldots, n\} : x_i \subseteq y_i) \Rightarrow [\gamma](x_1, \ldots, x_n) \subseteq [\gamma](y_1, \ldots, y_n).$$

In $\mathbf{V_4}$ we also need $\mathbf{o}$ for getting the same complete expressivity over the same class of persistent connectives [Mus95].[10]

In most cases, functional expressivity of a propositional language can be demonstrated by means of so-called normal forms in the language, which specifies the class of satisfying truth-value assignments in an obvious way. In this section we only discuss the language with complete expressivity for $\mathbf{V_4}$, $\mathbf{V_c}$, $\mathbf{V_t}$ and $\mathbf{V_2}$.

Definition 7 *If X is a set of formulas, then $\gamma X := \{\gamma F \mid F \in X\}$ for a given unary connective γ. If $X = \{F_1, \ldots, F_n\}$ is a non-empty finite set of formula then $\bigwedge X := F_1 \wedge \ldots \wedge F_n$ and $\bigvee X := F_1 \vee \ldots \vee F_n$.*[11]

A conjunct form is a formula of the form:

$$\bigwedge W \wedge \bigwedge {\sim} X \wedge \bigwedge -Y \wedge \bigwedge -{\sim} Z, \text{ such that } \quad W, X, Y, Z \subseteq \mathrm{At}_0 \tag{2}$$

A 4-conjunct form is a conjunct form as in 2 with $W \cup Y = X \cup Z = \mathrm{At}_0$ and $W \cap Y = X \cap Z = \emptyset$. A c-conjunct form is a 4-conjunct form as in 2 with

[9] The connective set $\{\sim, \wedge\}$ has complete expressivity over so-called *closed* persistent connectives in $\mathbf{V_c}$ [Ben84]. Closed connectives always obtain a classical value, $\{0\}$ or $\{1\}$, if all its arguments have classical values.

[10] This result for persistence gives us immediately an answer to the question for which class of formulas 2-satisfiability is the same as c-satisfiability: all the formulas which can be defined in terms of $\mathbf{u}$, $\sim$ and $\wedge$.

[11] Of course, this is not a well-defined formula, but because of commutativity of $\vee$ and $\wedge$ this choice is unique op to logical equivalence.

$W \cap X = \emptyset$. Analogously, a t-conjunct form is obtained by taking $Y \cap Z = \emptyset$ and for a 2-conjunct form we stipulate $Y = Z = \emptyset$.

A disjunct form is a formula of the form:

$$\bigvee W \vee \bigvee {\sim} X \vee \bigvee -Y \vee \bigvee -{\sim} Z \text{ such that } \quad W, X, Y, Z \subseteq \mathrm{At}_0. \qquad (3)$$

The notions of $*$-disjunct form are defined analogously. A disjunct form in $L(\sigma)$ is said to be a clause.

A *prenex formula* $F \in L(\sigma)$ has the form $Q_1 x_1 \ldots Q_n x_n G(x_1, \ldots, x_n, y_1, \ldots, y_m)$, where G is quantifier free and $Q_i \in \{\forall, \exists\}$. G is called the matrix of F and is denoted by *matrix(F)*.

Proposition 4 (Propositional Normal Form) *Every propositional formula is 4-equivalent to either a disjunction of 4-conjunct forms, $\bot$, $\mathbf{o}$ or $\mathbf{u}$. Analogously, such a formula is 3-equivalent to $\bot$, $\mathbf{u}$ or a disjunction of 3-conjunct forms, and t-equivalent to $\bot$, $\mathbf{o}$ or a disjunction of t-conjunct forms.*

In general, it is not possible to obtain precise predicate logical version of proposition 4. Most often, so-called prenex normal forms are used to define versions of the normal form result above for the predicate logical case.

Proposition 5 (Prenex Normal Form) *For every formula $F(x_1, \ldots, x_n) \in L(\sigma)$ there are prenex formulas $G(x_1, \ldots, x_n), H(x_1, \ldots, x_n) \in L(\sigma)$ such that*
(1) $F \equiv_4 G$, and $F \equiv_4 H$;
(2) $matrix(G) = \bigvee X$, X is a set of conjunct forms, $matrix(H) = \bigwedge Y$, Y is a set of disjunct forms.

3.3 Proof Theory

In this subsection we will present sequent calculi for partial logics. As mentioned earlier, other styles of derivation calculi are also possible. There are several reasons to chose for the sequential style. First, they make the axiomatic differences between different partial logics and classical logic immediately visible. Second, meta-theoretic proofs about the relations between deduction and model-theory, such as correctness and completeness proofs, benefit from a sequential proof theory. Third, in many cases sequential systems turn out to be shorter.[12] For example, general completeness results for functionally complete languages, can be easily be transformed to completeness proofs for poorer sublanguages.

Definition 8 (Sequent) *A* sequent *s is an expression of the form*

$$F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$$

where $F_i, G_j \in L(\sigma)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The body *of s, denoted by Bs, is given by $\{F_1, \ldots, F_m\}$, and the* head *of s, denoted by Hs, is given by $\{G_1, \ldots, G_n\}$.* $\mathrm{Seq}(\sigma)$ *denotes the class of all sequents s such that $Hs, Bs \subseteq L(\sigma)$.*

[12] In partial predicate logic, this advantage of sequential systems does not become sharply evident. A branch of partial logic, which surely benefits in this respect from sequential axiomatization is partial modal logic, as have been shown in Jaspars[Jas94].

Definition 9 (Model of a Sequent) *Let $\mathcal{I} \in \boldsymbol{I}_4$. Then,*

$$\mathcal{I} \models F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n \quad \text{iff} \quad \bigcap_{i \leq m} Sat_{\mathcal{I}}(F_i) \subseteq \bigcup_{j \leq n} Sat_{\mathcal{I}}(G_j)$$

For $S \subseteq \mathrm{Seq}$, $\mathrm{Mod}_*(S)$ and $S \models_* s$ are defined analogously as in Definition 3.

Definition 10 (Sequential inference) *A* sequential inference rule *R has the form*

$$\frac{s_1 \ldots s_n}{s_{n+1}},$$

with $s_i \in \mathrm{Seq}(\sigma)$ for all $i \in \{1, \ldots, n\}$. The elements of $\{s_1, \ldots, s_n\}$ are called the assumptions *of R, and s_{n+1} is called the* conclusion *of R. If $n = 0$, that is rules without assumptions, we say that R is* axiomatic*, and simply write s_1. A* sequential system $\mathbf{s}$ *is a set of sequential inference rules. Every conclusion of an axiomatic rule in $\mathbf{s}$ is said to be $\mathbf{s}$-derivable in 0-steps. If $m > 0$ then a sequent s is said to be $\mathbf{s}$-derivable in m steps if there exists a rule $\frac{s_1, \ldots, s_k}{s} \in \mathbf{s}$ such that for all $i \in \{1, \ldots, k\}$ the sequents s_i are $\mathbf{s}$-derivable in less than m steps. A sequent is called $\mathbf{s}$-derivable if it is $\mathbf{s}$-derivable in a certain finite number of steps. These sequents $X \Rightarrow Y$ are called $\mathbf{s}$-sequents, and we write $\vdash_s X \Rightarrow Y$.*

Below we will present sequential systems for the partial logics which have been discussed earlier. As usual, we distinguish structural rules from introduction rules. Structural rules are syntactically independent of the logic which we are axiomatizing. Introduction rules stipulate the meaning of logical functors in a proof-theoretic fashion. Logical functors are introduced both in the head of a sequent (L-introduction) and in the body of a sequent (R-introduction). Furthermore, we distinguish between rules which introduce a new compound proposition as being true and those which define the falsity of a new compound proposition which then appears in the scope of the strong negation $\sim$ within the conclusion of the rule.[13] Every introduction rule is specified by an abbreviation of the form $\mathrm{X}^{\mathrm{v}}\gamma$, where $\mathrm{X} \in \{\mathrm{L}, \mathrm{R}\}$ (left or right), $\mathrm{v} \in \{\mathrm{true}, \mathrm{false}\}$ and γ specifies the connective or quantifier which is introduced.

Below we give a presentation of the rules which are relevant for the axiomatization of partial logic. Instead of $X \cup \{F\}$ we write X, F.

[13] In [FLV92] so-called *quadrants* have been introduced, which can be understood as a kind of four-placed sequents: $X|X' \Rightarrow Y|Y'$. The truth-conditional reading of such a quadrant is that all models which verify all members of X and falsify all members of X', verify at least one member of Y or falsify at least one member of Y'. This approach makes falsity introduction possible within the derivational format and is therefore somewhat more elegant. If we wish to axiomatize $\sim$-free sublanguages, such a choice would even be necessary in order to obtain complete inference systems in a sequential fashion.

Structural Rules

$$F \Rightarrow F \qquad \text{START}$$

$$\frac{X \Rightarrow Y, X \subseteq X', Y \subseteq Y'}{X' \Rightarrow Y'} \qquad \text{MON}$$

$$\frac{X, F \Rightarrow Y \quad X' \Rightarrow F, Y'}{X, X' \Rightarrow Y, Y'} \qquad \text{CUT}$$

This set of structural rules will be called **struc**.

Truth Rules

$$\frac{X \Rightarrow F, Y}{X, {\sim}F \Rightarrow Y} \quad \text{L}^{\text{true}}{\sim} \qquad \frac{X, F \Rightarrow Y}{X \Rightarrow {\sim}F, Y} \quad \text{R}^{\text{true}}{\sim}$$

$$\frac{X, F, G \Rightarrow Y}{X, F \wedge G \Rightarrow Y} \quad \text{L}^{\text{true}}\wedge \qquad \frac{X \Rightarrow F, Y \quad X' \Rightarrow G, Y'}{X, X' \Rightarrow F \wedge G, Y, Y'} \quad \text{R}^{\text{true}}\wedge$$

$$X, \bot \Rightarrow Y \quad \text{L}^{\text{true}}\bot$$

$$X \Rightarrow \mathbf{o}, Y \quad \text{R}^{\text{true}}\mathbf{o}$$

$$\frac{X, F[t/x] \Rightarrow Y \quad (1)}{X, \forall x F \Rightarrow Y} \quad \text{L}^{\text{true}}\forall \qquad \frac{X \Rightarrow F[c/x], Y \quad (2)}{X \Rightarrow \forall x F, Y} \quad \text{R}^{\text{true}}\forall$$

(1) = t substitutable for x in F

(2) = c is a closed term not occurring in $X \cup Y$

Furthermore, $\text{L}^{\text{true}}-$ and $\text{R}^{\text{true}}-$ are the rules which evolve from substituting $-$ for $\sim$ in the rules $\text{L}^{\text{true}}{\sim}$ and $\text{R}^{\text{true}}{\sim}$, respectively. For the 0-ary connective $\mathbf{u}$ we have only one rule; the same as as for $\bot$: $\text{L}^{\text{true}}\mathbf{u} = X, \mathbf{u} \Rightarrow Y$. All these rules together are called **true**.

Falsity rules

$$\frac{X, F \Rightarrow Y}{X, {\sim\sim}F \Rightarrow Y} \quad \text{L}^{\text{false}}{\sim} \qquad \frac{X \Rightarrow F, Y}{X \Rightarrow {\sim\sim}F, Y} \quad \text{R}^{\text{false}}{\sim}$$

$$\frac{X, {\sim}F \Rightarrow Y \quad X', {\sim}G \Rightarrow Y'}{X, X', {\sim}(F \wedge G) \Rightarrow Y, Y'} \quad \text{L}^{\text{false}}\wedge \qquad \frac{X \Rightarrow {\sim}F, {\sim}G, Y}{X \Rightarrow {\sim}(F \wedge G), Y} \quad \text{R}^{\text{false}}\wedge$$

$$X \Rightarrow {\sim}\bot, Y \quad \text{R}^{\text{false}}\bot$$

$$X, {\sim}\mathbf{u} \Rightarrow Y \quad \text{L}^{\text{false}}\mathbf{u}$$

$$\frac{X, {\sim}F[c/x] \Rightarrow Y \quad (2)}{X, {\sim}\forall x F \Rightarrow Y} \quad \text{L}^{\text{false}}\forall \qquad \frac{X \Rightarrow {\sim}F[t/x], Y \quad (1)}{X \Rightarrow {\sim}\forall x F, Y} \quad \text{R}^{\text{false}}\forall$$

(1) and (2) as in **true** above.

$\text{R}^{\text{false}}\mathbf{o}$ is the same as $\text{R}^{\text{false}}\bot$ with $\bot$ replaced by $\mathbf{o}$. For $-$ we have the same rules as for $\sim$. Simply substitute $-F$ for the occurrences ${\sim}F$ in $\text{L}^{\text{false}}{\sim}$ and $\text{R}^{\text{false}}{\sim}$ and we obtain $\text{L}^{\text{false}}-$ and $\text{R}^{\text{false}}-$ respectively. The complete set of these falsity rules will be called **false**.

We define the following sequential systems:

$$\mathbf{2} \quad = \quad \mathbf{struc} \cup (\mathbf{true} \setminus \{\text{R}^{\text{true}}\mathbf{o}, \text{L}^{\text{true}}\mathbf{u}\})$$

$$
\begin{aligned}
\mathbf{c} &= \mathbf{struc} \cup (\mathbf{true} \setminus \{\mathrm{R}^{\mathrm{true}}{\sim}, \mathrm{R}^{\mathrm{true}}\mathbf{o}\}) \cup (\mathbf{false} \setminus \{\mathrm{R}^{\mathrm{false}}\mathbf{o}\}) \\
\mathbf{t} &= \mathbf{struc} \cup (\mathbf{true} \setminus \{\mathrm{L}^{\mathrm{true}}{\sim}, \mathrm{L}^{\mathrm{true}}\mathbf{u}\}) \cup (\mathbf{false} \setminus \{\mathrm{L}^{\mathrm{false}}\mathbf{u}\}) \\
\mathbf{4} &= (\mathbf{c} \cap \mathbf{t}) \cup \{\mathrm{L}^{\mathrm{v}}\mathbf{u}, \mathrm{R}^{\mathrm{w}}\mathbf{o} \mid \mathrm{v}, \mathrm{w} \in \{\mathrm{true}, \mathrm{false}\}\}
\end{aligned}
$$

Below we will present completeness results of these systems with respect to the corresponding model-theoretic consequence relations. This completeness only holds when we presuppose the absence of exact predicates within the underlying signature. If σ contains exact predicates, we need to strengthen the systems **c** and **4** with a straightforward compensation for the loss of $\{\mathrm{R}^{\mathrm{true}}{\sim}\}$. Let $L(\sigma_{ex})$ be the sublanguage of $L(\sigma)$ which consists of all the proposition that only contain exact predicates. The systems **c-ex** and **4-ex** evolve from adding the rule $\mathrm{R}^{\mathrm{true}}_{\mathrm{ex}}{\sim}$ to **c** and **4**, respectively. This additional rule has the following form:

$$
\frac{X, F \Rightarrow Y \quad F \in L(\sigma_{ex})}{X \Rightarrow {\sim}F, Y} \;\; \mathrm{R}^{\mathrm{true}}_{\mathrm{ex}}{\sim}
$$

Observation 1 *The differences between* **2***,* **c***,* **t***, and* **4** *can also be described by means of relativized versions of* contraposition*. In* **2** *we have that*

$$
\vdash_2 X \Rightarrow Y \quad \Longleftrightarrow \quad \vdash_2 {\sim}Y \Rightarrow {\sim}X
$$

This is a form of contraposition for strong negation. In all the other systems we obtain this contraposition rule at least for the weak negation. The systems **c** *and* **t** *have mixed versions of the rule of contraposition:*

$$
\begin{aligned}
\vdash_c X \Rightarrow Y &\iff \vdash_c {\sim}Y \Rightarrow -X \\
\vdash_t X \Rightarrow Y &\iff \vdash_t -Y \Rightarrow {\sim}X
\end{aligned}
$$

The following proposition presents the completeness of the sequential systems of the previous paragraph. In fact, for the logic whose underlying language is functionally complete, these results can be already obtained by means of the translation of definition 6.

Proposition 6 (Completeness) *Let* **s** *be* **4***,* **c***,* **t** *or* **2***, and let $*$ refer to the associated model class, 4, c, t or 2, respectively. If σ is a signature with no exact predicates, then for all finite sets $X, Y \subseteq L(\sigma)$ we have:*

$$
\vdash_s X \Rightarrow Y \quad \mathit{iff} \quad \models_* X \Rightarrow Y
$$

If σ contains exact predicates, then the completeness result only holds for **2** *and* **t***. For* **4** *and* **c***, we have*

$$
\vdash_{3-ex} X \Rightarrow Y \quad \mathit{iff} \quad \models_c X \Rightarrow Y \quad \mathit{and} \quad \vdash_{4-ex} X \Rightarrow Y \quad \mathit{iff} \quad \models_4 X \Rightarrow Y
$$

The partial results of soundness are the left-to-right directions of the equivalences in the above proposition. These results can be checked by a straightforward induction on the length of derivation.

In order to give an ordinary Henkin-style proof of these completeness theorems, we need to define the notion of *saturated sets*. This is a generalization of the notion of maximally consistent sets, which is needed to prove the completeness for partial logics with poorer expressivity. Especially, when the weak negation is lacking, the requirement of maximal consistency is too strong.

Definition 11 (Saturation) *Let* $\mathbf{s}$ *be a sequential inference system. A set* $X \subseteq L(\sigma)$ *is called* $\mathbf{s}$*-saturated iff for all finite sets* $X', Y' \subseteq L(\sigma)$ *and* $X' \subseteq X$*:*

$$\textit{If} \quad \vdash_s X' \Rightarrow Y' \quad \textit{then} \quad Y' \cap X \neq \emptyset. \tag{4}$$

A set $X \subseteq L(\sigma)$ *is called* $\mathbf{s}$*-term-saturated iff* X *is saturated and for every* $\exists x F \in X$ *there exists a constant* c *in* σ *such that* $F[x/c] \in X$*.*

Note that for every $\mathbf{s}$-saturated X there exists no finite $X' \subseteq X$ such that $\vdash_s X' \Rightarrow \emptyset$. This property captures the $\mathbf{s}$-consistency of X.[14] Taking Y' in 4 to be a singleton tells us that $\mathbf{s}$-saturated sets are closed under $\mathbf{s}$-deduction. If Y' has multiple elements, the definition tells us that every 'disjunctive' conclusion from X breaks down into at least one element of X. In other words, the information in X does not contain disjunctive uncertainty. Complete certainty is captured by the defintion of term-saturation.

A further relevant observation here is that if a sequential system $\mathbf{s}$ contains **struc** and a rule $\dfrac{X, F \Rightarrow Y}{X \Rightarrow \gamma F, Y}$, then $\mathbf{s}$-saturated sets are the same as maximally $\mathbf{s}$-consistent sets.

Lemma 7 (Generalized Lindenbaum Lemma) *Let* X *and* Y *be two finite subsets of the language* $L(\sigma)$*, and let* $\mathbf{s} \in \{\mathbf{2}, \mathbf{c}, \mathbf{t}, \mathbf{4}\}$*. If* $\not\vdash_s X \Rightarrow Y$*, then there exists a* $\mathbf{s}$*-saturated set* $Z \subseteq L(\sigma)$ *such that* $X \subseteq Z$ *and* $Y \cap Z = \emptyset$*.*

The standard Lindenbaum lemma can be obtained by taking $Y = \emptyset$ in the general formulation above. Because saturation is the same as maximal consistency for systems with the L-TRUE rule for negation, the classical result is the same as saying that every consistent set is a subset of a maximal consistent set.

The generalization of the classical Lindenbaum lemma is due to Aczel and Thomason . The generalization of the classical result evolved from independent succesful attempts to prove the completeness of intuitionistic predicate logic [Acz68], [Tho69].

Most often, the proof of the generalized Lindenbaum lemma is presented by making use of syntactic expressivity of the language that one works with. In fact, the set of rules **struc** is enough to obtain the result [Jas95]. If $\not\vdash_s X \Rightarrow Y$, and $\{F_i\}_{i \in I\!N}$ is an enumeration of the language, we define the following sequence of sets of formulas:

$$\begin{aligned} X_0 &= X \\ X_{n+1} &= \begin{cases} X_n \cup \{F_n\} & \text{if } \not\vdash_s X_n, F_n \Rightarrow Y \\ X_n & \text{otherwise.} \end{cases} \end{aligned}$$

[14] If $\vdash_s X' \Rightarrow \emptyset$ then $\vdash X' \Rightarrow F$ for all F by application of MON. Note that a sequential system $\mathbf{s}$ which contain the rules **struc** is conistent iff $\not\vdash_s \emptyset \Rightarrow \emptyset$.

The limit of this sequence is an $\mathbf{s}$-saturated set, which contains X and does not intersect Y.

In the completeness proofs of partial predicate logics, we need term-saturated sets instead of saturated sets. The (cheap) trick to obtain these term-saturated sets is to extend the language with a countably infinite number of additional constants (also called *parameters*). Let $L(\sigma')$ be such an extension of $L(\sigma)$, and let X and Y be two finite subsets of the latter language.

Corollary 8 *If $\not\vdash_s X \Rightarrow Y$ then there exists an $\mathbf{s}$-term-saturated $Z \subseteq L(\sigma')$ such that $X \subseteq Z$ and $Z \cap Y = \emptyset$.*

This result immediately follows from lemma 7 and by taking a unique fresh parameter as an instantiation for each existentially quantified formula to obtain the desired term-saturated set.

The following lemma, which is also called the *truth lemma*, tells us that a term-saturated set verifies exactly those formulas which it contains. To formulate this result properly, we associate with every $\mathbf{s}$-term-saturated set $X \subseteq L(\sigma)$ an interpretation $\mathcal{I}^{\mathbf{s}}_X$

$$\begin{aligned}
U^{\mathcal{I}^{\mathbf{s}}_X} &= \text{the set of all closed terms of } \sigma; \\
f^{\mathcal{I}^{\mathbf{s}}_X} &= f \text{ for all functions and constants } f; \\
P^{\mathcal{I}^{\mathbf{s}}_X} &= \{\vec{t} \mid P(\vec{t}) \in X\}; \\
\widetilde{P}^{\mathcal{I}^{\mathbf{s}}_X} &= \{\vec{t} \mid {\sim}P(\vec{t}) \in X\} \text{ for all predicates } P.
\end{aligned}$$

Lemma 9 (Truth Lemma) *Let $\mathbf{s}$ be a system which contains the rules* **struc**, *and let X be $\mathbf{s}$-term-saturated:*

$$\mathcal{I}^{\mathbf{s}}_X \models F \Leftrightarrow F \in X.$$

The proof of this lemma consists of a fairly straightforward induction on the construction of formulas. In fact every connective or quantifier only uses its own introduction rules. This settles the completeness result also for poorer languages over the different model classes.

The final argument of the completeness result is an immediate consequence of lemma 7, corollary 8 and lemma 9. Suppose that X and Y are finite subsets of $L(\sigma)$ and $\not\vdash_s X \Rightarrow Y$. According to corollary 8 there exists an $\mathbf{s}$-term-saturated set Z in a parametrized superlanguage $L(\sigma')$ such that $X \subseteq Z$ and $Y \cap Z = \emptyset$. Lemma 9 above tells us that $\mathcal{I}^{\mathbf{s}}_Z \models F$ for all $F \in X$ and $\mathcal{I}^{\mathbf{s}}_Z \not\models G$ for all $G \in Y$. In other words, $\not\models_* X \Rightarrow Y$ where $*$ refers to the associated model class.[15]

[15]It is not hard to verify that $\mathcal{I}^{\mathbf{s}}_Z \in \boldsymbol{I}_*(\sigma)$ for all $\mathbf{s}$-term-saturated sets Z.

4 Minimal Reasoning

In this section we study several versions of nonmonotonic reasoning based on partial logic. In the first subsection nonmonotonic reasoning is analysed in an abstract setting. This is done by using the concept of a deductive frame and its semantical counterpart, a model-theoretic frame. On this level of abstraction one can give a characterization of several kinds of partial propositional logic. The second subsection is devoted to Herbrand models. Several theorems are generalized to partial logics, in particular the proposition about canonical models of universal theory. In the third subsection minimal models are investigated. Then, a new class of models is introduced, the Φ-paraminimal models of a universal theory which are a generalization of the good models of [TEG93]. Subsection 4.4 concludes with an investigation of compactness properties of the introduced nonmonotonic model operators.

4.1 Inference Frames and Model-Theoretic Frames

Let L be a language and $C : 2^L \to 2^L$ an inference operation. A condition on C is said to be *pure* if it concerns the operation alone without regard to its interrelations to classical consequence operation and truth-functional connectives. The most important pure conditions are the following.

$$\begin{array}{lr} X \subseteq Y \subseteq C(X) \Rightarrow C(Y) \subseteq C(X) & \text{(Cut)} \\ X \subseteq Y \subseteq C(X) \Rightarrow C(X) \subseteq C(Y) & \text{(Cautious Monotony)} \\ X \subseteq Y \subseteq C(X) \Rightarrow C(X) = C(Y) & \text{(Cumulativity)} \\ C(C(X)) \subseteq C(X) & \text{(Idempotence)} \end{array}$$

An inference operation C is cumulative iff C satisfies inclusion, cut and cautious monotony. Besides the three conditions of cut, cautious monotony and cumulativity [Mak93] emphasizes several mixed conditions of inference: *supraclassicality*, *distributivity*, and *rationality*. C is said to be supraclassical if it extends the usual consequence operation Cn of classical logic, ie. $Cn(X) \subseteq C(X)$ for all $X \subseteq L$. Obviously, these mixed conditions can be formulated for any logic. [16] For this purpose we use the following definition [Her95]

Definition 12 *1.* (L, C_L, C) *is said to be an* inference frame *iff the following conditions are satisfied:*

(a) L *is a language.*

(b) C_L *is an inference operation on* L *satisfying inclusion, idempotence and monotony.*

(c) C *is an inference operation on* L *extending* C_L*, i.e.* $C_L(X) \subseteq C(X)$*.*

2. An inference frame (L, C_L, C) *satisfies*

[16] This point of view was assumed in [FL94]

(a) left absorption *iff* $C_L(C(X)) = C(X)$*;*

(b) congruence *or* right absorption *iff* $C_L(X) = C_L(Y) \Rightarrow C(X) = C(Y)$*;*

(c) full absorption *iff it satisfies left absorption and congruence.*

If full absorption holds, C_L *is called a* monotonic basis *for* C*.*

3. *An inference frame* (L, C_L, C) *is said to be a* deductive frame *if it is compact and satisfies full absorption. In this case,* C_L *is called a* deductive basis *for* C*.*

If C_L is compact then the system (L, C_L, C) is called a *compact inference frame.* A semantics of an inference frame can be introduced by a *model-theoretic frame.*

Definition 13 $(L, \boldsymbol{I}, \models, \Phi)$ *is a* model-theoretic frame *iff*

1. $(L, \boldsymbol{I}, \models)$ *is a model-theoretic system;*
2. $\Phi : 2^L \to 2^M$ *is a functor such that* $\Phi(X) \subseteq Mod_I(X)$*.* Φ *is called* model operator.

Every model operator Φ corresponds to an inference operation $C_\Phi(X) = Th(\Phi(X))$. C_Φ extends C_I and satisfies left absorption, and hence (L, C_I, C_Φ) is an inference frame.

A model operator Φ is said to be *invariant* with respect to a model-theoretic system $(L, \boldsymbol{I}, \models)$ iff for all $X \subseteq L$, $\Phi(X) = \Phi(C_I(X))$. A model-theoretic frame $(L, \boldsymbol{I}, \models, \Phi)$ is said to be compact if C_I satisfies compactness; it is called invariant if the model operator Φ is invariant wrt $(L, \boldsymbol{I}, \models)$.

Proposition 10 *If* Φ *is invariant for the compact model-theoretic system* $(L, \boldsymbol{I}, \models)$ *then* (L, C_I, C_Φ) *is a deductive frame.*

In order to obtain a semantics for a nonmonotonic inference system (L, C) we proceed in two steps: first we have to find an appropriate deductive basis (L, C_L, C); then we have to construct a model-theoretic semantics for the deductive system (L, C_L) which will finally yield a model-theoretic frame representing the deductive frame (L, C_L, C).

A set $X \subseteq L$ is said to be *deductively closed* iff $C_L(X) = X$. X is *deductively consistent* (in short, *d-consistent*) if $C_L(X) \neq L$. A deductive system (L, C_L) is called *explosive* iff there exists a finite subset $Y \subseteq L$ such that $C_L(Y) = L$. C_L is *negation explosive* if there is a unary functor $n : L \to L$ in the language such that for every $X \subseteq L$, and every $F \in L$, the following holds: $C_L(X \cup \{F\}) = L$ iff $n(F) \in C_L(X)$. A set $X \subseteq L$ is *maximally d-consistent* if $C_L(X) \neq L$ and for every proper superset Y of X it holds that $C_L(Y) = L$.

Observation 2 *The deductive systems* $(L_0(\sigma), C_*)$*, where* $* \in \{2, c, 4\}$*, are explosive and negation explosive.*

Proof: We consider only the case $* = c$, the other cases are analogous. Let F be an arbitrary sentence and $G := F \wedge -F$. Obviously, $C_c(G) = L_0(\sigma)$. To prove that C_c is negation explosive let $n(F) =_{df} -F$. In general we have $\mathrm{Mod}_c(X) = \emptyset$ if and only if $C_c(X) = L$. Let $C_c(X \cup \{F\}) = L$, then $\mathrm{Mod}_c(X \cup \{F\}) = \emptyset$. We prove, that $X \models_c -F$. Assume, $X \not\models_c -F$, then there is a coherent model $\mathcal{I} \models X$ such that $\mathcal{I} \not\models -F$, hence $\mathcal{I} \models F$. But then $\mathrm{Mod}_c(X \cup \{F\}) \neq \emptyset$, a contradiction. Conversely, assume $X \models_c -F$. It is sufficient to show that $\mathrm{Mod}_c(X \cup \{F\}) = \emptyset$. Assume $\mathrm{Mod}_c(X \cup \{F\}) \neq \emptyset$; then there is a coherent interpretation $\mathcal{I}$ such that $\mathcal{I} \models X, F$. From this follows $X \not\models_c -F$, a contradiction.

Proposition 11 *If (L, C_L) is explosive then every d-consistent subset of L can be extended to a maximally d-consistent set.*

Closed sets can be used to represent models, and to build model-theoretic semantics for deductive systems. Let (L, C_L) be a deductive system and $Cs(L) = \{X \subseteq L : C_L(X) = X\}$. For every subset $M \subseteq Cs(L)$ the following model-theoretic system $(L, M, \models)$ can be introduced. Define for $F \in L$ and $m \in M$: $m \models F$ iff $F \in m$. The model-theoretic system $(L, M, \models)$ represents a semantics for (L, C_L) iff $C_M = C_L$; then it is called a *Lindenbaum-Tarski-semantics* for (L, C_L). Obviously, a subset $M \subseteq Cs(L)$ represents a L-semantics for (L, C_L) iff for all consistent $X \subseteq L$ it holds that $C_L(X) = \bigcap(Cs(X) \cap M)$. This observation implies the following proposition.

Proposition 12 *A subset $M \subseteq Cs(L)$ represents a semantics for (L, C_L) if for every d-consistent subset $X \subseteq L$ and $F \notin C_L(X)$ there is an extension $X \subseteq m$, $m \in M$ such that $F \notin m$.*

For the construction of a semantics it is sufficient to select a subset of $Cs(L)$ representing the models. X is said to be *relatively maximal* (abbreviated *r-maximal*) iff there is a formula $F \in L$ such that $F \notin C_L(X)$ and for every proper superset Y of X the condition $F \in C_L(Y)$ is satisfied. Obviously, every r-maximal set is deductively closed. Let $rmax(L) \subseteq Cs(L)$ be the set of all relatively maximal subsets wrt (L, C_L).

Proposition 13 (Lindenbaum-Tarski) *Let (L, C_L) be a deductive system, $X \subseteq L$, and $F \notin C_L(X)$, then there exists a maximal extension $Y \supseteq X$, such that $F \notin Y$.*

Observation 3 *$rmax(L)$ is smallest subsystem of $Cs(L)$ representing a semantics for (L, C_L). We call it the Lindenbaum-Tarski standard semantics (LT-semantics).*

Definition 14 *The inference operations C_4, C_c, C_t, C_2 can be characterized as follows. We restrict our consideration to the case of propositional logic. Let $Ax_4(Prop)$ be the following set of formulas:*

1. $F \supset (G \supset F)$
2. $(F \supset (G \supset H)) \supset ((F \supset G) \supset (F \supset H))$
3. $(F \supset (G \supset H)) \supset (G \supset (F \supset H))$
4. $(F \supset G) \supset (-G \supset -F)$
5. $\sim - F \supset F$
6. $F \supset \sim - F$
7. $\sim\sim F \supset F$
8. $F \supset \sim\sim F$
9. $(F \wedge G) \supset F$
10. $(F \wedge G) \supset G$
11. $(F \supset (G \supset H)) \supset ((F \wedge G) \supset H)$
12. $(\sim F \supset\sim (F \wedge G)$
13. $(\sim G \supset\sim (F \wedge G)$
14. $((\sim F \supset H) \supset ((\sim G \supset H) \supset (\sim (F \wedge G) \supset H))$

$Ax_t(Prop) = Ax_4(Prop) \cup \{-F \supset \sim F/F \in Fm(Prop)\}$;
$Ax_c(Prop) = Ax_4(Prop) \cup \{\sim F \supset -F/F \in Fm(Prop)\}$;
$Ax_2(Prop) = Ax_c(Prop) \cup Ax_t(Prop)$.
Rules: Modus ponens : $\{(F, F \supset G/G) : F, G$ *formulas* $\}$.

Observation 4 (Completeness Theorem) *Let* $X \subseteq Fm(Prop)$ *and* $* \in \{2, c, 4, t\}$. $D_*(X)$ *is the smallest set containing* $X \cup Ax_\star(Prop)$ *and closed with respect to modus ponens. Define* $X \vdash_\star F$ *iff* $F \in D_*(X)$. *Then,*

$$X \models_\star F \quad \textit{iff} \quad X \vdash_\star F.$$

Proof (scetch for $\models_4$): A set X of formulas is said to be complete iff the following conditions are fulfilled:
$F \notin X$ iff $-F \in X$,
$F \wedge G \in X$ iff $\{F, G\} \subseteq X$,
$F \vee G \in X$ iff $\{F, G\} \cap X \neq \emptyset$,
$\sim -F \in X$ iff $F \in X$,
$\sim\sim F \in X$ iff $F \in X$,
$\sim (F \wedge G) \in X$ iff $\{\sim F, \sim G\} \cap X \neq \emptyset$,
$\sim (F \vee G) \in X$ iff $\{\sim F, \sim G\} \subseteq X$.
If X is complete then the set $\mathcal{I} = \{l \in Lit_0(\sigma) : l \in X\}$ is a partial model of X. To prove the completeness theorem we assume $X \models_4 F$ but $X \not\vdash_4 F$. By proposition 13 there is a maximal set $Y \supseteq X \cup Ax_4$ such that $Y \not\vdash_4 F$. It can be shown that Y is complete and deductively closed. This implies $F \in Y$, hence $-F \in Y$. Then there exists a model $\mathcal{I} \models Y$ such that $\mathcal{I} \not\models F$. This is a contradiction to $X \models_4 F$.$\Box$

Deductive frames can be semantically characterized as follows [DH94].

Proposition 14 *Let* $\mathcal{F} = (L, C_L, C)$ *be a deductive frame. Then there exists a model-theoretic frame* $\mathcal{S} = (L, M, \models, \Phi)$ *such that* Φ *is invariant and* $\mathcal{S}$ *represents* $\mathcal{F}$.

The subsequent schema summarizes the general method for constructing a semantics for a given inference system. The main point here is to find the right deductive basis in the set $\{C_L : (L, C_L, C) \text{ is a deductive frame }\}$. In many cases a deductive basis (L, C_L) can be chosen to be maximal [Die94].

$$(L, C)$$
$$\Downarrow$$
Construction of a deductive frame
$$\Downarrow$$
$$(L, C_L, C)$$
$$\Downarrow$$
Construction of a model-theoretic frame
$$\Downarrow$$
$$(L, M, \models, \Phi)$$
such that $C_L = C_M$, and $C = C_\Phi$

4.2 Herbrand Models

A *partial Herbrand interpretation* in the language $L(\sigma)$ is one for which the universe equals $U(\sigma)$, and the function symbols have their canonical interpretation. In this section we study model-theoretic frames based on Herbrand interpretations. Let $\boldsymbol{I}_*^H(\sigma)$ be the set of all Herbrand interpretations in $\boldsymbol{I}_*(\sigma)$, with $* \in \{4, c, t, 2\}$, and $\mathrm{Mod}_*^H(X) = \boldsymbol{I}_*^H \cap \mathrm{Mod}_*(X)$, $X \subseteq L(\sigma)$. The corresponding consequence relation $\models_*^H$ is defined by $X \models_*^H F \Leftrightarrow \mathrm{Mod}_*^H(X) \subseteq \mathrm{Mod}_*(F)$.

Definition 15 (Diagram) *The* diagram *of a σ-interpretation $\mathcal{I}$ is defined as $D_\mathcal{I} = \{l \in \mathrm{Lit}_0(\sigma) : \mathcal{I} \models l\}$.*[17]

Observation 5 *Partial Herbrand interpretations can be identified with their diagrams.*

Proof: Let $\mathcal{I} = (U(\sigma), (f^\mathcal{I})_{f \in Fun}, (R^\mathcal{I})_{R \in Rel})$ be a Herbrand interpretation and $t_1, \ldots, t_n \in U(\sigma)$. Then $\mathcal{I} \models R(t_1, \ldots, t_n)$ iff $\langle t_1^\mathcal{I}, \ldots, t_n^\mathcal{I} \rangle \in R^\mathcal{I}$ and $\mathcal{I} \models {\sim}R(t_1, \ldots, t_n)$ iff $\langle t_1^\mathcal{I}, \ldots, t_n^\mathcal{I} \rangle \in \tilde{R}^\mathcal{I}$. ¿From this follows that $D_\mathcal{I}$ represents the set $R^\mathcal{I} \cup \tilde{R}^\mathcal{I}$. $\Box$

Herbrand interpretations over σ can be considered as subsets of $\mathrm{Lit}_0(\sigma)$. Then the set $\boldsymbol{I}_4^H(\sigma)$ coincides with $2^{\mathrm{Lit}_0(\sigma)}$; $\boldsymbol{I}_c^H(\sigma) = \{\mathcal{J} \subseteq Lit_0(\sigma) :$ s.th. there is no $l \in At_0(\sigma)$ satisfying $\{l, {\sim}l\} \subseteq \mathcal{J}\}$; $\boldsymbol{I}_t^H(\sigma) = \{\mathcal{J} :$ for all $l \in At_0(\sigma) : \{l, {\sim}l\} \cap \mathcal{J} \neq \emptyset\}$; and $\boldsymbol{I}_2^H(\sigma) = \boldsymbol{I}_c^H(\sigma) \cap \boldsymbol{I}_t(\sigma)$. A consistent set $X \subseteq L(\sigma)$ does not always have a Herbrand model.

[17] Notice that, strictly speaking, we define the ground diagram, and not the full diagram.

Observation 6 *There are consistent sets* $X \subseteq L_0(\sigma)$ *without a Herbrand model:* $X = \{P(a), \forall x(P(x) \supset P(f(x)), \exists x(-P(x))\}$.

Let $\sigma = \langle Rel, ExRel, Const, Fun\rangle$ be a signature, $\mathcal{I}$ a partial σ-interpretation, and $U_1 \subseteq U_{\mathcal{I}}$. The *restriction of* $\mathcal{I}$ to U_1 is a partial interpretation $\mathcal{J}$, denoted by $\mathcal{J} = \mathcal{I} \downarrow U_1$, which is defined by the following conditions:
(1) the subset U_1 is closed with respect to the functions $\{f^{\mathcal{I}} : f \in Fun\}$, and $\{c^{\mathcal{I}} : c \in Const\} \subseteq U_1$;
(2) for every $R \in Rel \cup ExRel$: $R^{\mathcal{J}} = R^{\mathcal{I}} \cap U_1^{ar(R)}$ and $\widetilde{R}^{\mathcal{J}} = \widetilde{R}^{\mathcal{I}} \cap U_1^{ar(R)}$.
$\mathcal{J}$ is said to be a *substructure* of $\mathcal{I}$ if there is a subset $U_1 \subseteq U_{\mathcal{I}}$ such that $\mathcal{J} = \mathcal{I} \downarrow U_1$.

Proposition 15 *Let* $\forall x_1 \dots x_m A(x_1, \dots, x_m, y_1, \dots, y_n) = B(y_1, \dots, y_n) \in L(\sigma)$ *be a universal formula,*$A(\overline{x}, \overline{y})$ *quantifier free,* $\mathcal{I} \in \mathrm{Mod}_c(\sigma)$*, and* $\mathcal{I}, \mu \models B(y_1, \dots, y_n)$*,* μ *an evaluation and* $\mu(y_1) = a_1, \dots, \mu(y_n) = a_n$*. Let* $\mathcal{J}$ *be a substructure of* $\mathcal{I}$ *such that* $\{a_1, \dots, a_n\} \subseteq U_{\mathcal{J}}$*. Then* $\mathcal{J}, \mu \models B(y_1, \dots, y_n)$.

Proof: Assume $\mathcal{I}, \mu \models B(y_1, \dots, y_n)$, and denote this condition by the expression $\mathcal{I} \models B[a_1, \dots, a_n]$. Since $B[a_1, \dots, a_n]$ is universal it follows that for all $b_1, \dots, b_n \in U_{\mathcal{I}}$ the condition $\mathcal{I} \models A[b_1, \dots, b_m, a_1, \dots a_n]$ is satisfied. Because the formula $A(\overline{x}, \overline{y})$ does not contain quantifiers it follows $\mathcal{J} \models A[b_1, \dots, b_m, a_1, \dots a_n]$, provided $\{b_1, \dots, b_m, a_1, \dots a_n\} \subseteq U_{\mathcal{J}}$. This implies $\models B[a_1, \dots, a_n]$. $\Box$.

Corollary 16 *Let* $\mathcal{I} \in \boldsymbol{I}_c(\sigma)$*,* $F \in L_0(\sigma)$ *a universal sentence, and* $\mathcal{J} \subseteq \mathcal{I}$ *a substructure of* $\mathcal{I}$*. Then* $\mathcal{I} \models F$ *implies* $\mathcal{J} \models F$.

Proposition 17 *Let* $S \subseteq L(\sigma)$ *be a universal theory of signature* σ *and* $Const(\sigma) \neq \emptyset$*. If* S *has a coherent model then it has a coherent Herbrand model.*

Proof: Let $\mathcal{I}$ be a model of $\mathrm{Mod}_c(S)$. A Herbrand model $\mathcal{I}_0$ is defined as follows.
(1) $U(\mathcal{I}_0) = U(\sigma)$.
(2) $\langle t_1, \dots, t_n\rangle \in R^{\mathcal{I}_0}$ iff $\mathcal{I} \models R(t_1, \dots, t_n)$ and $\langle t_1, \dots, t_n\rangle \in \widetilde{R}^{\mathcal{I}_0}$ iff $\mathcal{I} \models {\sim}R(t_1, \dots, t_n)$, where $R \in Rel(\sigma)$.
From (2) follows for every quantifier free formula $A(x_1, \dots, x_n)$ and terms $t_1, \dots, t_n \in U(\sigma)$:
(3) $\mathcal{I}_0 \models A(t_1, \dots, t_n)$ iff $\mathcal{I} \models A(t_1, \dots, t_n)$.
Now, let $A \in S$, and $A = \forall x_1 \dots x_k G(x_1, \dots, x_k)$. Assume, $\mathcal{I}_0 \not\models \forall x_1 \dots x_k G(\overline{x})$, then there is an evaluation ν such that $\mathcal{I}_0, \nu \not\models \forall \overline{x} G(\overline{x})$. By definition this is equivalent to the existence of variable free terms $t_1, \dots, t_n$ such that $\mathcal{I}_0 \models -G(t_1, \dots, t_n)$; by condition (3) this is equivalent to $\mathcal{I} \not\models G(t_1, \dots, t_n)$. But then $\mathcal{I} \not\models \forall x_1 \dots x_n G(\overline{x})$ which is a contradiction to the assumption. $\Box$.

Observation 7 *The relation* $\models_c^H$ *is not axiomatizable, i.e. there are decidable sets* $X \subseteq L(\sigma)$ *such that* $\{F : X \models_c^H F\}$ *is not recursively enumerable.*

Proof: Let PA be the axioms of Peano Arithmetic in the signature $\sigma = (0, +, \circ, s)$; then $PA \models_c^H F$ iff F is true in the standard model of arithmetic. This gives a contradiction to Gödel's incompleteness theorem.□

Proposition 18 *Let S be a universal theory, and $F = \exists\overline{x}G(\overline{x})$ a closed existential formula. Then $S \models_c^H G$ iff $S \models_c F$.*

Proof: The implication $(\leftarrow)$ is trivial. We show $(\rightarrow)$. Assume $S \models_c^H F$, but $S \not\models_c \exists\overline{x}G(\overline{x})$. then there is a partial model $\mathcal{I} \in \mathrm{Mod}_c(S)$ such that $\mathcal{I} \not\models \exists\overline{x}G(\overline{x})$, and hence $\mathcal{I} \models \forall\overline{x} - G(\overline{x})$. Then $S \cup \{\forall\overline{x} - G(\overline{x})\}$ has a model and by proposition there is a Herbrand model $\mathcal{I}_0$ for $S \cup \{\forall\overline{x} - G(\overline{x})\}$. Since $\mathcal{I}_0 \models S$ this implies $S \not\models_c^H F$, a contradiction. □.

Proposition 18 cannot be generalized to universal sentences.

Observation 8 *For every language $L(\sigma)$, σ containing a relational symbol of arity ≥ 1, there exists a universal theory $S \subseteq L(\sigma)$ and a universal sentence F such that $S \models_c^{H(\sigma)} F$ but $S \not\models_c F$.*

Proof: W.l.o.g., we assume that σ contains a unary relational symbol $P(x)$. Let $S = \{P(t) : t \in U(\sigma)\}$, then $S \models_c^H \forall xP(x)$, but, obviously, $S \not\models_c \forall xP(x)$.□

Definition 16 (Persistent Formula) *A formula $F \in L(\sigma)$ is called* persistent *if for arbitrary partial Herbrand interpretations $\mathcal{I}, \mathcal{J}$ over σ satisfying $\mathcal{I} \subseteq \mathcal{J}$, and every substitution $\theta : Var \rightarrow U_{\mathcal{I}}$ the condition $\mathcal{I} \models F\theta$ implies $\mathcal{J} \models F\theta$.*

Observation 9 *Every formula $F \in L(\sigma; \sim, \wedge, \vee, \exists, \forall)$ is persistent.*

Proof: (inductively on the complexity of F). Let $l \in \mathrm{Lit}(\sigma)$ and $\mathcal{I} \models l\theta$; then $l\theta \in \mathcal{I}$ and hence $l\theta \in \mathcal{J}$ for every extension $\mathcal{J} \geq \mathcal{I}$. Let $\mathcal{I} \models (G \vee H)\theta$; then $\mathcal{I} \models G\theta$ or $\mathcal{I} \models H\theta$. By induction hypothesis it holds $\mathcal{J} \models G\theta$ or $\mathcal{J} \models H\theta$, and hence $\mathcal{I} \models (G \vee H)\theta$. Similarly, this is proved for $F = G \wedge H$.

Now let be $F = \exists\overline{x}G(\overline{x}, \overline{y})$ and $\lambda : Var \rightarrow U(\sigma)$ is a substitution such that $\mathcal{I} \models G(\lambda(\overline{x}), \theta(\overline{y}))$. By induction hypothesis $\mathcal{J} \models G(\lambda(\overline{x}), \theta(\overline{y}))$, and this implies $\mathcal{J} \models \exists\overline{x}G(\overline{x}, \theta(\overline{y}))$. Finally, $F = \forall\overline{x}G(\overline{x}, \theta(\overline{y}))$. Then, for every substitution $\lambda : \overline{x} \rightarrow U(\mathcal{I})$ $\mathcal{I} \models G(\lambda(\overline{x}), \theta(\overline{y}))$. By induction hypothesis $\mathcal{I}_1 \models G(\lambda(\overline{x}), \theta(\overline{y}))$, and since $U_{\mathcal{I}} = U_{\mathcal{J}}$ it follows $\mathcal{J} \models \forall\overline{x}G(\overline{x}, \theta(\overline{y}))$. □.

Proposition 19 *Let S be a universal theory, and $F = \exists\overline{x}G(\overline{x})$ a closed existential sentence. Then the following conditions are equivalent:*
(1) $S \models_c F$.
(2) There are variable free substitutions $\theta_1, \ldots, \theta_n$ such that $S \models_c \bigvee_{i \leq n} G(\theta_i(\overline{x}))$.

Proof: Assume $S \models_c F$; since F is an existential sentence this is equivalent to $S \models_c^H F$. This is the case if and only if for every Herbrand model $\mathcal{I}$ of S there is a substitution $\theta_\mathcal{I}$ such that $\mathcal{I} \models_c G\theta_\mathcal{I}$. From this follows that $S \models_c^H \bigvee\{G\theta_\mathcal{I} : \mathcal{I}$ is an Herbrand model of $S\}$. By the compactness theorem for $\mathcal{L}_c$ there is a finite set Δ of Herbrand models of S such that $S \models_c \bigvee\{G\theta_\mathcal{I} : \mathcal{I} \in \Delta\}$. □

Proposition 19 can also be proved for $\models_t$ and $\models_4$. For $\models_2$ this proposition is Herbrand's theorem.

4.3 Minimal Models

In the sequel we introduce several versions of minimal models; we assume that all interpretations under consideration are Herbrand interpretations.

Definition 17 (Extension) *Let $\mathcal{I}$ and $\mathcal{I}'$ be two interpretations. We say that $\mathcal{I}'$* extends *$\mathcal{I}$, symbolically $\mathcal{I} \leq \mathcal{I}'$, if $D_\mathcal{I} \subseteq D_{\mathcal{I}'}$.*

This ordering of interpretations corresponds to the intuitive notion of *information growth*. It has also been called *knowledge ordering* in the literature.

Definition 18 (Minimally Inconsistent Models) *Let $Inc(\mathcal{I}) = D_\mathcal{I} \cap \widetilde{D_\mathcal{I}}$ measure the inconsistency of a four-valued interpretation $\mathcal{I}$. Then*

$$\mathrm{Mod}^H_{mi}(X) = \{\mathcal{I} \in \mathrm{Mod}^H_4(X) : \neg\exists \mathcal{I}' \in \mathrm{Mod}^H_4(X),\ s.th.\ Inc(\mathcal{I}') \subset Inc(\mathcal{I})\}$$

is the class of minimally inconsistent models of $X \subseteq L(\sigma)$.

Minimally inconsistent models were introduced in [Pri89]. Like plain four-valued models they tolerate inconsistency, but they are, in a sense, logically more conservative as the following example shows.

Example 1 (Disjunctive Syllogism) *Four-valued inference does not respect the Disjunctive Syllogism, but minimally inconsistent inference does:*

$$\{p \vee q, {\sim}q\} \not\models_4 p,\ but\ \{p \vee q, {\sim}q\} \models_{mi} p.$$

Notice that whenever $X \subseteq L$ has a coherent model, then $\mathrm{Mod}_{mi}(X) = \mathrm{Mod}_c(X)$, i.e. $\models_c$ can be viewed as a restriction of $\models_{mi}$ to coherent knowledge bases.

Definition 19 (Minimal Models) *Let $X \subseteq L(\sigma)$, and $* \in \{c, 4, mi\}$. Then $\mathrm{Mod}^m_*(X) = \mathrm{Min}(\mathrm{Mod}^H_*(X))$ is the class of all minimal $*$-models of X with respect to $\leq$. Similarly, $\mathrm{Mod}^{max}_*(X) = \mathrm{Max}(\mathrm{Mod}^H_*(X))$ is the class of all maximal $*$-models of X.*

The following systems are important model-theoretic frames: $(L, \boldsymbol{I}^H_*, \models, \mathrm{Mod}^m_*)$, where $* \in \{c, 4\}$, and L is a sublanguage of $L(\sigma)$, and furthermore $(L, \boldsymbol{I}^H_4, \models, \mathrm{Mod}^m_{mi})$.

Observation 10 *There are theories $T \subseteq L(\sigma)$ which are c-satisfiable, i.e.* $\mathrm{Mod}_c(T) \neq \emptyset$*, but do not have minimal models:* $\mathrm{Mod}_c^m(T) = \emptyset$.

Proof: Let $T_<$ be the theory of linear ordering with first but without last element; P is a unary predicate satisfying the following property: $\exists x P(x) \wedge \forall v \forall u (P(u) \wedge v > u \supset P(v))$, i.e. P is a nonempty cofinal segment of the linear ordering. Then every partial model of this theory is not minimal.

Observation 11 *Let $\boldsymbol{K} \subseteq \boldsymbol{I}_4^H$. An interpretation $\mathcal{I} \in \boldsymbol{I}_4^H$ is said to be minimal in $\boldsymbol{K}$ if $\mathcal{I} \in \boldsymbol{K}$, and there is no $\mathcal{J} \in \boldsymbol{K}$ such that $\mathcal{J} < \mathcal{I}$. Then the following holds: An interpretation $\mathcal{I} \in \boldsymbol{I}_4^H$ is minimal in $\boldsymbol{I}_t^H$ if $\mathcal{I}$ is 2-valued.*

From the results of section 3 the following observation can be easily derived.

Observation 12 *For every set S of universal sentences there is a set of clauses $Cl(S)$ such that* $\mathrm{Mod}_*(S) = \mathrm{Mod}_*(Cl(S))$*, $* \in \{4, c, t\}$.*

Proposition 20 *Let S be a universal theory in $L(\sigma)$. Every partial model from $\boldsymbol{I}_c^H$ of S is an extension of a minimal coherent model of S and can be extended to a maximal coherent model of S.*

Proof: Let S be given; we may assume that S is a set of clauses. Let $\mathcal{I}$ be a coherent model of S and $\Omega(\mathcal{I}) = \{\mathcal{J} : \mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \models S\}$. We show that every decreasing chain $\mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \ldots \supseteq \mathcal{I}_n \ldots$ in $(\Omega(\mathcal{I}), \subseteq)$ has a lower bound. Using Zorn's lemma this implies the existence of a minimal element, which is a minimal partial coherent model of S. Assume $\mathcal{I}^\star = \bigcap_{n \in \omega} \mathcal{I}_n$, and $\mathcal{I}_n \models S$ for every $n \in \omega$. We show that $\mathcal{I}^\star \models S$. Choose $C \in S$, and

$C = E_1 \vee \ldots \vee E_k \vee \sim F_1 \vee \ldots \sim F_l \vee -G_1 \vee \ldots \vee -G_m \vee -\sim H_1 \vee \ldots \vee -\sim H_n$,

where $E_p, F_q, G_r, H_s \in At(\sigma)$. Assume $\mathcal{I}^\star \not\models C$; this is the case if and only if

$$\mathcal{I}^\star \models -\forall \overline{x}(\bigvee_{p \leq k} E_p \vee \bigvee_{q \leq l} \sim F_q \vee \bigvee_{r \leq m} -G_r \vee \bigvee_{s \leq n} -\sim H_s).$$

implying that $\mathcal{I}^\star \models \exists \overline{x}(\bigwedge_{p \leq k} -E_p \wedge \bigwedge_{q \leq l} -\sim F_q \wedge \bigwedge_{r \leq m} - - G_r \wedge \bigwedge_{s \leq n} -(-\sim H_s))$ which is equivalent to

$$\mathcal{I}^\star \models \exists \overline{x}(\bigwedge_{p \leq k} -E_p \wedge \bigwedge_{q \leq l} -\sim F_q \wedge \bigwedge_{r \leq m} G_r \wedge \bigwedge_{s \leq n} \sim H_s).$$

There is an evaluation $\theta : Var \to U(\sigma)$, such that

$$\mathcal{I}^\star \models \bigwedge_{p \leq k} -E_p\theta \wedge \bigwedge_{q \leq l} -\sim F_q\theta \wedge \bigwedge_{r \leq m} G_r\theta \wedge \bigwedge_{s \leq n} \sim H_s\theta.$$

¿From this follows that $(\{E_p\theta\} \cup \{\sim F_q\theta\}) \cap \mathcal{I}^\star = \emptyset$. This implies the existence of a number $m \in \omega$ such that $\mathcal{I}_m \cap (\{E_p\theta\} \cup \{\sim F_q\theta\}) = \emptyset$. On the other hand, since $\mathcal{I}^\star \models \bigwedge_{r \leq m} G_r\theta \wedge \bigwedge_{s \leq n} \sim H_s\theta$, then by persistence of formulas without weak negation for every extension $\mathcal{J} \supseteq \mathcal{I}^\star$ it holds $\mathcal{J} \models \bigwedge G_r\theta \wedge \bigwedge \sim H_s\theta$. Alltogether, we may conclude $\mathcal{I}_m \models \bigwedge_{p\leq k} -E_p\theta \wedge \bigwedge_{q\leq l} -\sim F_q\theta \wedge \bigwedge_{r\leq m} G_r\theta \wedge \bigwedge_{s\leq t} \sim H_s\theta$. But then $\mathcal{I}_m \not\models \forall\overline{x}C$, and this is a contradiction. The proof for the existence of maximal models is analogous. $\square$.

Proposition 20 holds also for 4-valued and for total models. Let $\mathrm{Mod}_*^{max}(T)$ be the set of maximal $*$-models of T $, * = c, 4$.

Proposition 21 *Let S be a universal theory in $L(\sigma)$, and $\Delta^+(S) = \{l \in Lit_0(\sigma) : S \models_* l\}$, $\Delta^-(S) = \{l \in Lit_0(\sigma) : S \models_* -l\}$. Then:*
(1) $\bigcap \mathrm{Mod}_*^m(S) = \Delta^+(S)$.
(2) $Lit_0(\sigma) - \bigcup \mathrm{Mod}_*^{max}(S) = \Delta^-(S)$.

Proof: (1) Let $l \in \bigcap \mathrm{Mod}_*^m(S)$, then $l \in \mathcal{I}$ for every $\mathcal{I} \in \mathrm{Mod}_*^H(S)$, since every $\mathcal{I} \in \mathrm{Mod}_*^H(S)$ is an extension of some $\mathcal{J} \in \mathrm{Mod}_*^m(S)$. Hence, $l \in \Delta_+(S)$. If $l \in \Delta^+(S)$, then $l \in \mathcal{I}$ for every $\mathcal{I} \in \mathrm{Mod}_*^H(S)$ and it follows $\mathcal{I} \in \bigcap \mathrm{Mod}_*^m(S)$.

(2) Let $l \in Lit_0(\sigma) - \bigcup \mathrm{Mod}_*^{max}(S)$, then for every $\mathcal{I} \in \mathrm{Mod}_*^H(S)$: $l \notin \mathcal{I}$, since $\mathcal{I}$ can be extended to a maximal model $\mathcal{J} \in \mathrm{Mod}_*^H(S)$. It follows that $l \in \Delta^-(S)$. Now, let $l \in \Delta^-(S)$, then $S \models -l$. This implies $l \notin \mathcal{I}$ for every model $\mathcal{I} \models S$ and in particular $l \notin \bigcup \mathrm{Mod}_*^{max}(S)$, hence $l \in Lit_0(\sigma) - \bigcup \mathrm{Mod}_*^{max}(S)$. $\square$

Definition 20 (Paraminimal Models) *Let $X \subseteq L(\sigma)$, $* = c, 4, mi$, and $\boldsymbol{K} \subseteq \mathrm{Mod}_*^H(X)$. Then,*

$$\mathrm{Mod}_*^m(\boldsymbol{K}, X) = \mathrm{Min}(\{\mathcal{I} \in \mathrm{Mod}_*^H(X) : \bigcup \boldsymbol{K} \subseteq \mathcal{I}\})$$

is the set of all minimal $*$-supermodels of $\boldsymbol{K}$. The set $\mathrm{Mod}_*^{pm}(\boldsymbol{K}, X)$ of paraminimal $*$-models over $\boldsymbol{K}$ is the smallest set of $*$-models of X containing $\boldsymbol{K}$ and being closed with respect to the condition:

$$(\alpha) \quad \textit{if} \quad \boldsymbol{M} \subseteq \mathrm{Mod}_*^{pm}(\boldsymbol{K}, X) \quad \textit{then} \quad \mathrm{Mod}_*^m(\boldsymbol{M}, X) \subseteq \mathrm{Mod}_*^{pm}(\boldsymbol{K}, X).$$

If in condition (α) the set $\boldsymbol{M}$ is assumed to be finite then the resulting set, denoted by $\mathrm{Mod}_*^{fpm}(\boldsymbol{K}, X)$, is the set of finitely based paraminimal $*$-models over $\boldsymbol{K}$. Finally, the set of paraminimal $*$-models of X is defined by $\mathrm{Mod}_*^{pm}(X) = \mathrm{Mod}_*^{pm}(\mathrm{Mod}_*^m(X), X)$, and the set of finitely based paraminimal $*$-models by $\mathrm{Mod}_*^{fpm}(X) = \mathrm{Mod}_*^{fpm}(\mathrm{Mod}_*^m(X), X)$.

The paraminimal model operator is the basis of the following model-theoretic frames: $(L, \boldsymbol{I}_*^H, \models, \mathrm{Mod}_*^{pm})$, where $* \in \{c, 4\}$, and $(L, \boldsymbol{I}_4^H, \models, \mathrm{Mod}_{mi}^{pm})$. Let $(L, M, \models, \Phi)$ be a model-theoretic frame based on a partial logic $\mathcal{L}_*$. The set of Φ-paraminimal models of X, denoted by $\mathrm{Mod}_*^{pm}(\Phi, X)$, is defined by $\mathrm{Mod}_*^{pm}(\Phi(X), X)$. We introduce following notation: $C_*^{fpm}(X) = Th(\mathrm{Mod}_*^{fpm}(\mathrm{Mod}_*^m(X), X))$. Obviously, $C_c^{pm}(X) \subseteq C_c^{fpm}(X) \subseteq C_c^m(X)$.

Our notion of a paraminimal $*$-model is a generalization of the 'good models' defined in [TEG93] for classical theories. In the next section we will combine the idea of paraminimality with the idea of stability which is essential for an adequate interpretation of nonpersistent sequents, resp. generalized logic programming rules.

Paraminimal models can be classified with respect to a rank notion. We set $\mathrm{Mod}_*^{pm}(0,X) = \emptyset$; $\mathrm{Mod}_*^{pm}(1,X) = \mathrm{Mod}_*^{m}(X)$; and for $\alpha \geq 1$,

$$\mathrm{Mod}_*^{pm}(\alpha+1,X) = \mathrm{Mod}_*^{pm}(\alpha,X) \cup \bigcup\{\mathrm{Mod}_*^{m}(\boldsymbol{K},X) : \boldsymbol{K} \subseteq \mathrm{Mod}_*^{pm}(\alpha,X)\}$$

and finally for limit ordinals,

$$\mathrm{Mod}_*^{pm}(\lambda,X) = \bigcup_{\beta<\lambda} \mathrm{Mod}_*^{pm}(\beta,X)$$

A paraminimal model $\mathcal{I} \in \mathrm{Mod}_*^{pm}(X)$ has rank α, denoted by $rk(\mathcal{I}) = \alpha$, iff $\mathcal{I} \in \mathrm{Mod}_*^{pm}(\alpha+1,X) - \mathrm{Mod}_*^{pm}(\alpha,X)$. The *p-rank* of X, abbreviated $prk(X)$, is defined by $prk(X) = \sup\{rk(\mathcal{I}) : \mathcal{I} \in \mathrm{Mod}_*^{pm}(X)\}$.

Example 2 *Let $T = \{a \vee b \vee c \vee d,\ a \wedge b \supset c \wedge d \wedge e \wedge f,\ c \wedge d \supset e \vee f\}$. Then the largest paraminimal model of T is $abcdef$; since it is the minimal supermodel of the two minimal models a and b it has rank 1. There are exactly two paraminimal models of rank 2: $cdef$ and $bcdef$, consequently $prk(T) = 2$.*

Observation 13 *Let $X \subseteq Prop(\sigma)$ contain persistent formulas only. Then $prk(X) \leq 1$.*

Proof: Let $\mathrm{Min}_*(\mathcal{I})$ be the set of all minimal submodels of $\mathcal{I}$, and $\boldsymbol{K}$ be a set of submodels of $\mathcal{I}$ being models of X. If $\mathcal{I}$ is a minimal supermodel of $\boldsymbol{K}$ then by the persistence of X it holds that $\mathcal{I} = \bigcup \boldsymbol{K}$. We show that the rank hierarchy stabilizes at 1, i.e. $\mathrm{Mod}_*^{pm}(1,X) = \mathrm{Mod}_*^{pm}(2,X)$. Let $\mathcal{I} \in \mathrm{Mod}_*^{pm}(2,X)$, then there is a set $\boldsymbol{M}$ of submodels of $\mathcal{I}$ such that $\boldsymbol{M} \subseteq \mathrm{Mod}_*^{pm}(1,X)$ and $\mathcal{I}$ is a minimal supermodel of $\boldsymbol{M}$. By the above remark $\mathcal{I} = \bigcup \boldsymbol{M}$. Furthermore, every $\mathcal{J} \in \boldsymbol{M}$ can be represented by $\mathcal{J} = \bigcup \mathrm{Min}_*(\mathcal{J})$. From this follows that $\mathcal{I} = \bigcup \mathrm{Min}(\mathcal{I})$, i.e. $rk(\mathcal{I}) = 1$.

If Y is a partially ordered set, then we can select those elements from Y which are minimal upper bounds of certain minimal elements of Y by means of an operator

$$\mathrm{PMin}^1(Y) = \{X \in Y \mid \neg\exists X' \in Y : X' < X \ \&\ \mathrm{Min}_{X'}(Y) = \mathrm{Min}_X(Y)\}$$

where $\mathrm{Min}_X(Y) = \{X' \in \mathrm{Min}(Y) : X' \leq X\}$. We obtain the following corollary.

Corollary 22 *Let $X \subseteq Prop(\sigma)$ contain persistent formulas only. Then,*

$$\mathrm{Mod}_*^{pm}(X) = \mathrm{PMin}^1(\mathrm{Mod}_*^{H}(X))$$

Eventually, an important question is: which of the inference relations $\models_y^x$ for $x = m, pm$, and $y = 4, c, mi$, is the natural choice for knowledge systems. We shall see below that the answer to this questions depends also on the logical expressiveness of the language of knowledge bases. In the simplest case, where only *extensional* knowledge, corresponding to sentences from $L(\sim, \wedge, \vee)$, is represented the preferred inference relation is based on paraminimal models, i.e. $\models_{mi}^{pm}$ (resp. $\models_c^{pm}$ if only consistent KBs are admitted), as the following example illustrates.

Example 3 (Inclusive Disjunction) *Let $X = \{q(c),\ p(a) \vee p(b)\}$. From this KB we want to be able to infer $-p(c)$, but not $-p(a) \vee -p(b)$. However, $X \not\models_* -p(c)$, for $* = c, mi$, but $X \models_*^m -p(c)$, since*

$$\mathrm{Mod}_*^m(X) = \{\{q(c), p(a)\},\ \{q(c), p(b)\}\}$$

and also, $X \models_*^m -p(a) \vee -p(b)$, which is not wanted. Therefore, we need paraminimal reasoning:

$$\mathrm{Mod}_*^{pm}(X) = \{\{q(c), p(a)\},\ \{q(c), p(b)\},\ \{q(c), p(a), p(b)\}\}$$

and hence, $X \not\models_*^{pm} -p(a) \vee -p(b)$.

4.4 Compactness Properties

We conclude this section with the investigation of compactness properties. Let $\mathcal{F} = (L, \boldsymbol{I}, \models \Phi)$ be a model-theoretic frame. C_Φ is *semantically compact* if for every set $X \subseteq L$ the following holds: if $\Phi(X_f) \neq \emptyset$ for every finite subset $X_f \subseteq X$ then $\Phi(X) \neq \emptyset$. In classical logic compactness and semantical compactness coincide. For arbitrary model-theoretic frames this is not longer true. The following facts clarify the relation between compactness and semantical compactness. Φ is *strongly semantical compact* iff for every set $X \subseteq L$ and formula $\phi \in L$ the following holds: if $\Phi(X_f) \cap \mathrm{Mod}(\phi) \neq \emptyset$ for every finite subset $X_f \subseteq X$ then $\Phi(X) \cap \mathrm{Mod}(\phi) \neq \emptyset$. The following proposition shows the interrelation between these properties.

Proposition 23 *Let $\mathcal{F} = (L, \boldsymbol{I}, \models, \Phi)$ be a model-theoretic frame.*

1. *Assume (L, C_I) is explosive. If C_Φ is compact then it is semantically compact.*
2. *Assume (L, C_I) is negation explosive. Then C_Φ is strongly compact if and only if it is compact.*

Let C be an inference operation on the language L, and C_f be the *finitary restriction* of C, i.e. $dom(C) = \{X : X \subseteq L,\ X \text{ is finite }\}$, and $C_f(X) = C(X)$ for all finite subsets X of L. Let C be monotonic, and $\Delta_0(C_f)(X) = \bigcup_{Y \in Fin(X)} C_f(Y)$. Δ_0 can be considered as an operator extending finitary inference operation to infinitary ones, and if C is monotonic then $\Delta_0(C_f) \leq C$. If C is monotonic and compact then $\Delta_0(C_f) = C$, ie. C is uniquely defined by its finitary restriction via Δ_0. In case C is not compact, but monotonic, $\Delta_0(C)$ gives an approximation of C from below. If C does not satisfy monotony then there is no well-defined operator Δ allowing to reconstruct the operation C from its finitary restriction C_f. To analyse this phenomenon we use the following notions from [Her95].

Definition 21 *Let (L, C_L) be a deductive system. $\mathcal{D}(L, C_L) = \{C : (L, C_L, C)$ is a deductive frame $\}$; $\mathcal{D}_f(L, C_L) = \{C : C$ is finitary and (L, C_L, C) is a deductive frame $\}$; $\mathcal{I}(L, C_L) = \{C : (L, C_L, C)$ is an inference frame $\}$.*

1. *A functor* $\Delta : \mathcal{D}_f(L, C_L) \to \mathcal{I}(L, C_L)$ *is said to be an extension operator if for every* $C \in \mathcal{D}(L, C_L)$ *the conditions* $dom(\Delta(C)) = 2^L$ *and* $\Delta(C) \downarrow Fin(L) = C$ *are satisfied.* Δ *is called deductive if* $im(\Delta) \subseteq \mathcal{D}(L, C_L)$.

2. *An inference operation* $C : 2^L \to 2^L$ *is* Δ*-compact iff* $C \subseteq \Delta(C_f)$; C *is completely* Δ*-compact iff* $C = \Delta(C_f)$.

Abstract compactness properties can be expressed by conditions $compcond(C_L, C\ , Fin(L))$ depending on C, C_L and the finite subsets of the language L. Important compactness properties are summarized in the following definition [DH94].

Definition 22 *Let* (L, C_L, C) *be a deductive frame.*

1. C *is weakly compact iff for every* $X \subseteq L$, $\phi \in C(X)$ *there is a finite subset* $A \subseteq C_L(X)$ *such that* $\phi \in C(A)$.

2. C *is weakly supracompact iff for every* $X \subseteq L$, $\phi \in C(X)$ *and every finite* $A \subseteq C_L(X)$ *there is a finite set* B, $A \subseteq B \subseteq C_L(X)$ *such that* $\phi \in C(B)$.

3. *Let* F *be an inference operation defined for finite sets only.* $\Delta_{wsc}(F)(X) = \{\phi :$ *for every finite* $A \subseteq C_L(X)$ *there is a finite* B *such that* $A \subseteq B \subseteq C_L(X)$ *and* $\phi \in F(B)\}$.

The concepts in the preceding definition are modifications and generalizations of compactness notions introduced and studied in [FL94]. The operator Δ_{wsc} was introduced and presented in [DH94]. In the following we show that the extension operator Δ_{wsc} is suitable for analysing minimal reasoning in partial propositional and partial predicate logic.

The set $Prop(\sigma)$ of propositional sentences over σ is defined by $Prop(\sigma) = L_0(\sigma : \{\wedge, \vee, \sim, -\})$. Let $V \subseteq Lit_0(\sigma)$ and $Prop(V)$ the smallest set of formulas in $L(\sigma)$ containing V and closed with respect to $\wedge, \vee, \sim, -$. Obviously, $Prop(\sigma) = Prop(Lit_0(\sigma))$. Given $F \in Prop(\sigma)$ then $lit(F) =$ the set of literals from $Lit_0(\sigma)$ appearing in F, and $lit(X) = \bigcup\{lit(F) : F \in X\}$. To simplify the notation let $\mathrm{Mod}(X)$ be the set of all coherent Herbrand models of X, $X \subseteq Prop(\sigma)$. For a set $V \subseteq Lit_0(\sigma)$ let $\mathrm{Mod}_V(X) = \{\mathcal{I} \cap V : \mathcal{I} \in \mathrm{Mod}(X)\}$. The deductive frame under consideration is defined by $(Prop(\sigma), \boldsymbol{I}_c^H, \models, \mathrm{Mod}_c^m)$.

Proposition 24 *Let* $V \subseteq Lit_0(\sigma)$, $F \in Prop(V)$, *and* $\mathcal{I} \in \boldsymbol{I}_c^H$. *Then* $\mathcal{I} \models F$ *if and only if* $\mathcal{I} \cap V \models F$.

Proof: We may assume that F is in negation form. The proof is inductively on the complexity of F. We consider only the case $F = -A$. Let $\mathcal{I} \models -A$, then $A \notin \mathcal{I}$, hence $A \notin \mathcal{I} \cap V$, this implies $\mathcal{I} \cap V \models -A$. Conversely, let $\mathcal{I} \cap V \models -A$, then $A \notin \mathcal{I} \cap V$; by assumption $A \in V$, hence $A \notin \mathcal{I}$, and this implies $\mathcal{I} \models A$, hence $\mathcal{I} \models -A$. The remaining cases are straightforward. □

Proposition 25 *If* $V \subseteq Lit_0(\sigma)$, $F \in Prop(V)$, *then* $X \models F$ *if and only if* $\mathrm{Mod}_V(X) \subseteq \mathrm{Mod}(\{F\})$.

Proposition 26 *Let $X \subseteq Prop(\sigma)$, $V \subseteq lit(X)$ a finite subset. Then there is a finite subset $B \subseteq C_c(X)$, such that $lit(B) = V$ and $\mathrm{Mod}_V(X) = \mathrm{Mod}_V(B)$.*

Proof: $\mathrm{Mod}_V(X) = \{\mathcal{I} \cap V : \mathcal{I} \in \mathrm{Mod}(X)\}$ is a finite set of cardinality $\leq 2^{card(V)}$. For every $\mathcal{J} \in \mathrm{Mod}_V(X)$ let $d(\mathcal{J}) = \bigwedge \mathcal{J} \wedge \bigwedge\{-l : l \in V - \mathcal{J}\}$, and $F = \bigvee\{d(\mathcal{J}) : \mathcal{J} \in \mathrm{Mod}_V(X)\}$. Then $B = \{F\}$ satisfies the desired condition.□

Obviously, the model operator Mod_c^m is semantically compact, since for every set X the condition $\mathrm{Mod}_c(X) \neq \emptyset$ implies $\mathrm{Mod}_c^m(X) \neq \emptyset$. In [PW90] it is shown that C_c^m is not deductively compact. The following simpler example is due to *J. Dietrich*. Let $Lit_0(\sigma)$ be infinite, and $\{p_i : i \in \omega\}$ an enumeration of $Lit_0(\sigma)$. The set X is defined as follows $X = \{p_1 \wedge \ldots \wedge p_i \wedge (p_{i+1} \vee p_0) : 1 \leq i < \omega\}$. Then $X \models_c^m -(p_0 \leftrightarrow p_1)$. If $\mathcal{I} \in \mathrm{Mod}_c^m(X)$ then $\mathcal{I} \models -p_0, m \models p_1$, hence $\mathcal{I} \models -(p_0 \leftrightarrow p_1)$. For every finite subset $X_f \subseteq X$ holds $X_f \not\models_c^m -(p_0 \leftrightarrow p_1)$.

Proposition 27 *The deductive frame $(Prop(\sigma), C_c, C_c^m)$, is weakly supracompact.*

Proof Let $X \models_c^m F$, $A \subseteq C_c(X)$, A finite and $lit(A) \cup lit(F) = \{l_1, \ldots, l_s\} = V$. By propositionh there is a finite subset $B \subseteq C_c(X)$ such that $lit(B) = \{l_1, \ldots, l_s\}$ and $\mathrm{Mod}_V(X) = \mathrm{Mod}_V(A \cup B)$. Let $\mathcal{J} \in \mathrm{Mod}_V^m(A \cup B)$, then $\mathcal{J} \subseteq V$. $\mathcal{J}$ can be extended to a model $\mathcal{I} \supseteq \mathcal{J}, \mathcal{I} \in \mathrm{Mod}(X)$. By proposition 20 there is a minimal model $\mathcal{I}_1 \in \mathrm{Mod}_V^m(X)$ such that $\mathcal{I}_1 \subseteq \mathcal{I}$. By assumption $\mathcal{I}_1 \models F$. It is $\mathcal{J} \subseteq \mathcal{I}_1$, since $\mathcal{J}$ is a minimal model of $A \cup B$. Then $\mathcal{J} \models F$ iff $\mathcal{I}_1 \models F$, hence $A \cup B \models_c^m F$.
□

Proposition 28 *Let $X \subseteq Prop(\sigma)$ and $F \in Prop(\sigma)$. Following conditions are equivalent:*

1. *$X \models_c^m F$,*

2. *for every finite subset $A \subseteq C_c(X)$ there exists a finite subset $B \subseteq C_c(X)$ such that $lit(B) \subseteq lit(A)$ and $A \cup B \models_c^m F$.*

Proof: The implication (1) $\rightarrow$ (2) follows immediately from proposition 27. We show (2) $\Rightarrow$ (1). Using the preciding propositions we construct a sequence $A_1, A_2, \ldots,$ of finite sets $A_i \subseteq C_c(X)$ such that $lit(F) \subseteq lit(A_1)$, $lit(A_i) \subseteq lit(A_{i+1})$, $lit(\bigcup_{i\in\omega}) = lit(X)$, and $\mathrm{Mod}_{lit(A_i)}(A_i) = \mathrm{Mod}_{lit(A_i)}(X)$. Denote $l(i) = lit(A_i)$. Obviously, $C_c(\bigcup_{i\in\omega} A_i) = C_c(X)$. By assumption for every A_i there is a $B_i \subseteq C_c(X)$ such that $lit(B_i) \subseteq lit(A_i)$, and $A_i \cup B_i \models_c^m F$. It is $\mathrm{Mod}_{l(i)}(A_i) = \mathrm{Mod}_{l(i)}(A_i \cup B_i)$, and since $lit(B_i) \subseteq lit(A_I)$ it follows $\mathrm{Mod}(A_i) = \mathrm{Mod}(A_i \cup B_i)$. This implies $A_i \models_c^m F$, $i = 1, 2, \ldots$.

We show that $X \models_c^m F$. Assume that this is not the case. Then there is a $\mathcal{I} \in \mathrm{Mod}_c^m(X)$ such that $\mathcal{I} \not\models F$. Then $\mathcal{I} \cap l(i)$ is not minimimal for A_i for every $i \in \omega$. Let $\mathcal{I}_i = \mathcal{I} \cap l(i)$ and $\Omega(i) = \{\mathcal{J} : \mathcal{J} \in \mathrm{Mod}_{l(i)}(A_i)$ and $\mathcal{J}$ is minimal for A_i, $\mathcal{J} \subseteq \mathcal{I}_i, \mathcal{J} \neq \mathcal{I}_i\}$. Obviously, $\mathcal{J} \models F$ for every $\mathcal{J} \in \Omega(i)$, $i \in \omega$. By assumption, the sets $\Omega(i)$ are nonempty for e every $i \in \omega$. For each $k \in \omega$ let

$\Lambda(k) = \{\mathcal{J} \cap l(k) : \mathcal{J} \in \bigcup_{j \geq i} \Omega(j)\}$. Let be $\Lambda = \bigcup_{k \geq 1} \Lambda(k)$. For $\mathcal{J} \in \Lambda$ let be $dom(\mathcal{J}) = l(k)$ iff $\mathcal{J} \in \Lambda(k)$ and for $\mathcal{J}_1, \mathcal{J}_2 \in \Lambda$: $\mathcal{J}_1 \sqsubseteq \mathcal{J}_2$ if $dom(\mathcal{J}_1) \subseteq dom(\mathcal{J}_2)$ and $\mathcal{J}_1 = \mathcal{J}_2 \cap dom(\mathcal{J}_1)$. Then $(\Lambda, \sqsubseteq)$ is a tree of finite valency. Furthermore, if $\mathcal{J} \in \Lambda(k)$ and $j < k$ then $\mathcal{J} \cap l(j) \in \Lambda(j)$, hence $\mathcal{J} \cap l(j) \sqsubseteq \mathcal{J}$. By König's lemma there is an infinite branch $\mathcal{B}$ in $(\Lambda, \sqsubseteq)$, and let $\mathcal{K} = \bigcup \mathcal{B}$. Obviously, $\mathcal{K} \models F$, because for every $\mathcal{J} \in \Omega(i)$; $\mathcal{J} \models F$, and $\mathcal{J} \cap l(j) \models F$ for every $j < i$. ¿From this follows $\mathcal{K} \neq \mathcal{I}$. Furthermore, $\mathcal{K} \models X$, since $\mathcal{K} \models A_i$, for every $i \in \omega$. We show that $\mathcal{K} \subseteq \mathcal{I}$. Assume this is not the case. Then there is a $v \in XB(\sigma)$ such that $v \in \mathcal{K}$ but $v \notin \mathcal{I}$. Then there is a $i \in \omega$ such that $v \in l(i)$. By construction, there is a $\mathcal{J} \in \Omega(j), j \geq i$, such that $\mathcal{K} \cap l(i) = \mathcal{J} \cap l(i)$; but $\mathcal{J} \subseteq \mathcal{I} \cap l(j)$. This gives a contradiction. It follows $\mathcal{K} \subseteq \mathcal{I}$, which is a contradiction to the minimality of $\mathcal{I}$. $\square$.

Proposition 29 (Corollary) *Let $(Prop(\sigma), C_c, C_c^m)$ be the deductive frame of minimal reasoning in partial propositional logic of coherent models. Then C_c^m is completely Δ_{wsc}-compact, ie. $C_c^m = \Delta_{wsc}((C_c^m)_f)$.*

5 Sequents and Stable Models

Traditionally, Gentzen sequents are used in a schematic way in sequent calculi, such as in 3.3, in order to express valid transitions from one argument schema to another. In other words, a sequent in a sequential inference rule stands for a whole class of propositional substitution instances.

In this section, we propose to use sequents in a non-schematic way for the purpose of representing rule knowledge. A sequent here is not a schematic but a concrete expression representing some piece of knowledge.

We define the following classes of sequents.

1. $\mathrm{Seq}_1(\sigma) = \{s \in \mathrm{Seq}(\sigma) \mid Bs, Hs \subseteq \mathrm{Lit}(\sigma)\}$.
2. $\mathrm{Seq}_2(\sigma) = \{s \in \mathrm{Seq}(\sigma) \mid Hs \subseteq \mathrm{Lit}(\sigma),\ Bs \subseteq \mathrm{XLit}(\sigma)\}$.
3. $\mathrm{Seq}_3(\sigma) = \{s \in \mathrm{Seq}(\sigma) \mid Hs \subseteq L(\sigma; \sim, \wedge, \vee),\ Bs \subseteq L(\sigma; -, \sim, \wedge, \vee, |, \supset)\}$.
4. $\mathrm{Seq}_4(\sigma) = \{s \in \mathrm{Seq}(\sigma) \mid Hs \subseteq L(\sigma; \sim, \wedge, \vee, \exists, \forall),\ Bs \subseteq L(\sigma; -, \sim, \wedge, \vee, |, \supset, \exists, \forall)\}$.

We also define $S' = \{s \in S \mid card(Hs) = 1\}$ for every class of sequents S. For $S \subseteq \mathrm{Seq}(\sigma)$, and $* = 4, c, t, 2$, we define the model operators

$$\begin{aligned}
\mathrm{Mod}_*(S) &= \{\mathcal{I} \in \boldsymbol{I}_*(\sigma) : \mathcal{I} \models s \text{ for all } s \in S\} \\
\mathrm{Mod}_*^H(S) &= \{\mathcal{I} \in \boldsymbol{I}_*^H(\sigma) : \mathcal{I} \models s \text{ for all } s \in S\} \\
\mathrm{Mod}_{mi}^H(S) &= \{\mathcal{I} \in \mathrm{Mod}_4^H(S) : \neg\exists \mathcal{I}' \in \mathrm{Mod}_4^H(S) \text{ s.th. } \mathrm{Inc}(\mathcal{I}') \subset \mathrm{Inc}(\mathcal{I})\}
\end{aligned}$$

and their minimal reasoning refinements

$$\begin{aligned}
\mathrm{Mod}_*^m(S) &= \mathrm{Min}(\mathrm{Mod}_*^H(S)) \\
\mathrm{Mod}_*^{pm}(S) &= \mathrm{Mod}_*^{pm}(\mathrm{Mod}_*^m(S), S)
\end{aligned}$$

The associated inference relations are defined as follows:

$$S \models^x_y F \quad \text{iff} \quad \mathrm{Mod}^x_y(S) \subseteq \mathrm{Mod}_y(F)$$

where $x = H, m, pm$, and $y = 4, c, t, 2, mi$, and $F \in L(\sigma)$.

Observation 14 *Let $B \Rightarrow H$ be any sequent. Then, for any $\mathcal{I} \in \boldsymbol{I}_4$,*

$$\mathcal{I} \models B \Rightarrow H \;\; \textit{iff} \;\; \mathcal{I} \models \bigwedge B \supset \bigvee H$$

This observation seems to imply that there is no big difference between sequents and material implications, since for $F, G \in L$, it holds that

$$\mathrm{Mod}_*(F \Rightarrow G) = \mathrm{Mod}_*(F \supset G)$$

However, for other model operators, such as stable models Mod^{ms}_* (see below), this is not the case.

Example 4 *Sequents differ from material implication:*

$$\mathrm{Mod}^{ms}_c(-p \supset q) = \{\{p\}, \{q\}\} \neq \mathrm{Mod}^{ms}_c(-p \Rightarrow q) = \{\{q\}\}$$

Observation 15 *Let $S \subseteq \mathrm{Seq}$ be a set of sequents. Then,*

$$\mathrm{Mod}^H_*(S) = \mathrm{Mod}^H_*([S])$$

where $[S]$ is the Herbrand instantiation of S.

Observation 16 *Let $S \subseteq \mathrm{Seq}_3(\sigma)$, and $F \in L_0(\sigma)$ be a closed existential sentence. Then,*

$$S \models_* F \;\; \textit{iff} \;\; [S] \models_* F$$

5.1 Paraminimal Models for Persistent Sequents

A sequent $s \in \mathrm{Seq}_3$ is called *persistent*, if all body formulas $F \in Bs$ are persistent. For instance, all sequents from Seq_1 are persistent. For a set S of persistent sequents, its paraminimal models, $\mathrm{Mod}^{pm}_*(S)$, are the intended models, and thus $\models^{pm}_c$ (resp. $\models^{pm}_{mi}$) are the natural inference relations for consistent (resp. inconsistent) knowledge bases consisting of persistent sequents.

Example 5 *Let $S = \{\Rightarrow q(b);\ \Rightarrow p(a), p(b);\ \ p(x) \Rightarrow \sim q(x)\}$. Since*

$$\mathrm{Mod}^{pm}_{mi}(S) = \mathrm{Mod}^{pm}_c(S) = \{\{q(b), p(a), \sim q(a)\}\}$$

we obtain for $* = c, mi$

$$S \models^{pm}_* \sim q(a) \wedge -p(b)$$

Observation 17 *For a sequent set $S \subseteq \mathrm{Seq}'_1$, where the head of a sequent consists of a single literal, and its body of a set of literals, the notions of minimal and of paraminimal models coincide, and there is a unique minimal model, denoted $\mathcal{M}_S$. Formally,*

$$\mathrm{Mod}_4^{pm}(S) = \mathrm{Mod}_4^m(S) = \{\mathcal{M}_S\}$$

Proof: We have to show that the interpretation $\mathcal{M}_S = \bigcap \mathrm{Mod}_4^H(S)$ is a model of S. Obviously, if it is a model, it is the least one.

Let $(B \Rightarrow l) \in [S]$ and $\mathcal{M}_S \models B$. By persistence of B we have $\mathcal{M}' \models B$ for every $\mathcal{M}' \in \mathrm{Mod}_4^H(S)$. This implies that $l \in \mathcal{M}'$ for every $\mathcal{M}' \in \mathrm{Mod}_4^H(S)$, and hence $l \in \mathcal{M}_S$. □

5.2 Stable Models for Non-Persistent Sequents

When a knowledge base consists of a set of sequents $S \subseteq \mathrm{Seq}_3$, where body formulas may be non-persistent, it may have (para)minimal models which are not intended. This is illustrated by the following example.

Example 6 (Local Closed-World Assumption)
Let $S = \{\Rightarrow q(c);\ \Rightarrow p(a), p(b);\ -p(x) \Rightarrow {\sim}p(x)\}$. The last sequent, from $-p(t)$ conclude ${\sim}p(t)$ for any term t, expresses a local Closed-World Assumption which is only admissible for exact predicates, i.e. $p \in ExRel$. Since we want to infer ${\sim}p(c)$, the following paraminimal models are not intended models:

$$\begin{aligned} M_1 &= \{q(c), p(c), p(a), {\sim}p(b)\} \\ M_2 &= \{q(c), p(c), p(b), {\sim}p(a)\} \\ M_3 &= \{q(c), p(c), p(a), p(b)\} \end{aligned}$$

Therefore, we need a more refined preference criterion which allows to select the intended models of a set of sequents from its Herbrand models.

Definition 23 $[\mathcal{M}_1, \mathcal{M}_2] = \{\mathcal{M} \in \boldsymbol{I}_4^H : \mathcal{M}_1 \leq \mathcal{M} \leq \mathcal{M}_2\}$

Recall that wrt a class of interpretations $\boldsymbol{K}$, we write $\boldsymbol{K} \models F$ iff $\mathcal{I} \models F$ for all $\mathcal{I} \in \boldsymbol{K}$. We denote the set of all sequents from a sequent set S which are applicable in $\boldsymbol{K}$ by

$$S_K = \{s \in [S] : \boldsymbol{K} \models Bs\}$$

The following definition of a *stable model* is inspired by the definition of a *stable closure* of a set of rules in [Wag94a].

Definition 24 (Stable Model) *Let $* = c, 4$. $\mathcal{M} \in \mathrm{Mod}_*^H(S)$ is called a* minimally stable $*$-model *of $S \subseteq \mathrm{Seq}_3(\sigma)$, symbolically $\mathcal{M} \in \mathrm{Mod}_*^{ms}(S)$, if there is a chain of Herbrand interpretations $\mathcal{M}_0 \leq \ldots \leq \mathcal{M}_\kappa$ such that $\mathcal{M} = \mathcal{M}_\kappa$, and*

1. *$\mathcal{M}_0 = \emptyset$.*

2. *For successor ordinals α with $0 < \alpha \leq \kappa$, $\mathcal{M}_\alpha$ is a minimal extension of $\mathcal{M}_{\alpha-1}$ satisfying the heads of all sequents whose bodies hold in $[\mathcal{M}_{\alpha-1}, \mathcal{M}]$, i.e.*

$$\mathcal{M}_\alpha \in \mathrm{Min}\{\mathcal{I} \in \boldsymbol{I}_*^H : \mathcal{I} \geq \mathcal{M}_{\alpha-1},\ \mathit{and}\ \mathcal{I} \models \bigvee Hs,\ \mathit{f.a.}\ s \in S_{[\mathcal{M}_{\alpha-1},\mathcal{M}]}\}$$

3. *For limit ordinals $\lambda \leq \kappa$,*

$$\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$$

Paraminimally stable coherent models are defined accordingly (replacing in the definition all occurences of 'minimal', resp. 'Min', by 'paraminimal', resp. 'PMin[1]'). The set of minimally stable $*$-models of S is denoted by $\mathrm{Mod}_*^{ms}(S)$, and the set of paraminimally stable models of S by $\mathrm{Mod}_*^{pms}(S)$. A further interesting class of models is defined by $\mathrm{Mod}_*^{pm}(\mathrm{Mod}_*^{ms}(S), S)$.

Minimally inconsistent stable models are defined by

$$\mathrm{Mod}_{mi}^*(S) = \{\mathcal{I} \in \mathrm{Mod}_4^*(S) : \neg\exists\mathcal{I}' \in \mathrm{Mod}_4^*(S) \text{ s.th. } \mathrm{Inc}(\mathcal{I}') \subset \mathrm{Inc}(\mathcal{I})\}$$

where $* = ms, pms$.

Example 6 (continued) *Only the following three paraminimal models of S are stable:*

$$\begin{aligned} M_4 &= \{q(c), \sim p(c), p(a), \sim p(b)\} \\ M_5 &= \{q(c), \sim p(c), p(b), \sim p(a)\} \\ M_6 &= \{q(c), \sim p(c), p(a), p(b)\} \end{aligned}$$

and hence, $S \models_c^{pms} \sim p(c)$.

Thus, $\models_c^{pms}$ (resp. $\models_{mi}^{pms}$) will be our preferred inference relation for knowledge-based reasoning.

Example 7 (Default Rules) *A default (resp. exception tolerant) rule can be expressed by a combination of weak and strong negation. E.g., the rule '*birds (normally) fly*' is expressed as*

$$b(x) \wedge -\sim f(x) \Rightarrow f(x)$$

If the knowledge base S contains in addition the facts that Tweety and Opus are birds, $b(T) \wedge b(O)$, but Opus does not fly, $\sim f(O)$, we can infer by stable reasoning that Tweety flies:

$$S \models_{mi}^{pms} f(T)$$

Paraminimally stable reasoning supports inclusive disjunctive information as the following example shows.

Example 8 (Inclusive Disjunction) *Let* $S = \{\Rightarrow p \vee q\,;\ -(p \wedge q) \Rightarrow r \vee s\}$. *Then,*

$$\begin{aligned}
\mathrm{Mod}_*^m(S) &= \{pr, ps, qr, qs, pq\} \\
\mathrm{Mod}_*^{pm}(S) &= \{pr, ps, qr, qs, pq, prs, qrs, pqr, pqs, pqrs\} \\
\mathrm{Mod}_*^{ms}(S) &= \{pr, ps, qr, qs\} \\
\mathrm{Mod}_*^{pms}(S) &= \{pr, ps, qr, qs, pq, prs, qrs\}
\end{aligned}$$

Stable models do not exist in all cases. For instance, $S = \{-p \Rightarrow p\}$ has exactly one minimal model, $\mathrm{Mod}_*^m(S) = \{\{p\}\}$, which is not stable, however. A sequent set, resp. logic program, without stable models will be called *unstable*.

Example 9 $S = \{p \supset q \Rightarrow r;\ r \Rightarrow p\}$ *is unstable.*

Observation 18 *Stable reasoning is not cumulative.*

Proof: The following counterexample is due to [vG88]. Let $S = \{-r \Rightarrow q\,;\ -q \Rightarrow r\,;\ -p \Rightarrow p;\ -r \Rightarrow p\}$. Since $\mathrm{Mod}_*^{ms}(S) = \{\{p, q\}\}$, and $S \models_*^{ms} p, q$, but $\mathrm{Mod}_*^{ms}(S \cup \{p\}) = \{\{p, q\},\ \{p, r\}\}$, and hence $S \cup \{p\} \not\models_*^{ms} q$. □

5.3 Extended Logic Programs as Sequent Sets

A sequent set $S \subseteq \mathrm{Seq}_2'$ corresponds to an *extended logic program (ELP)*

$$\Pi_S = \{l \leftarrow B : (B \Rightarrow l) \in S\}$$

The other way around, an extended logic program Π corresponds to a sequent set $S_\Pi \subseteq \mathrm{Seq}_2'$ with

$$S_\Pi = \{B \Rightarrow l : (l \leftarrow B) \in \Pi\}$$

For $B \subseteq \mathrm{XLit}(\sigma)$, let B^- denote the set of literals which occur weakly negated in B, i.e. $B^- := \{l \in \mathrm{Lit}(\sigma) : -l \in B\}$, and let $B^+ = \{l \in \mathrm{Lit}(\sigma) : l \in B\}$. It holds that for any $B \subseteq \mathrm{XLit}_0$, and any $\mathcal{I} \in \boldsymbol{I}_4^H$,

$$\mathcal{I} \models B \quad \text{iff} \quad B^+ \subseteq D_{\mathcal{I}}\ \&\ B^- \cap D_{\mathcal{I}} = \emptyset$$

Definition 25 (Immediate Consequence Operator) *Let* Π *be an extended logic program, and* $I \subseteq \mathrm{Lit}$ *be the diagram of* $\mathcal{I} \in \boldsymbol{I}_4^H$. *Then*

$$T_\Pi(I) = \{l \in \mathrm{Lit}_0 : \exists (l \leftarrow B) \in [\Pi],\ s.th.\ \mathcal{I} \models B\}$$

is called the immediate consequence operator *associated with* Π.

Definition 26 (Gelfond-Lifschitz 1990) *Let* $M \subseteq \mathrm{Lit}$, *and* Π *be an ELP. Then the* Gelfond-Lifschitz transformation *of* Π *with respect to* M *is defined as*

$$\Pi^M = \{l \leftarrow B^+ : (l \leftarrow B) \in [\Pi],\ and\ B^- \cap M = \emptyset\}$$

M *is called an* answer set *of* Π, *if* $\mathrm{Mod}_c^m(\Pi^M) = \{\mathcal{M}\}$, *and* $M = D_{\mathcal{M}}$.

We shall show below that the definition of answer sets is just a specialization of our notion of a stable model. The same holds for the definition of stable models of normal logic programs in [?]. Since these definitions are based on the Gelfond-Lifschitz-transformation Π^M requiring a specific rule syntax they are not very general; as a consequence, Gelfond and Lifschitz are not able to treat negation-as-failure as a logical functor, and to allow for arbitrary formulas in the body of a rule. The interpretation of negation-as-failure as weak negation in partial logic according to our stable semantics seems to be the first general logical treatment of nonmonotonic logic programs.[18] It was already proposed by Wagner in [Wag91, Wag94b], but without the full generality of the stable semantics proposed in the present paper.

Proposition 30 *An answer set of an extended logic program Π is the diagram of a minimally stable coherent model of the corresponding sequent set S_Π.*

Proof sketch: Let $M \subseteq \mathrm{Lit}$ be an answer set of an extended logic program Π, i.e. $\mathrm{Mod}_c^m(\Pi^M) = \{\mathcal{M}\}$, where $M = D_{\mathcal{M}}$. For $\Pi_M = \{l \leftarrow B \in [\Pi] : \mathcal{M} \models B\}$, the immediate consequence operator T_{Π_M} generates $\mathcal{M}$ as the supremum of the following chain:

$$M_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup T_{\Pi_M}(\bigcup_{\beta<\alpha} M_\beta))$$

It is easy to see for all rules $l \leftarrow B \in [\Pi]$, that $\mathcal{M}_\alpha \models l$ whenever $[\mathcal{M}_{\alpha-1}, \mathcal{M}] \models B$: simply because $l \in T_{\Pi_M}(\bigcup_{\beta<\alpha} M_\beta)$ whenever $\bigcup_{\beta<\alpha} M_\beta \models B$. It is also clear that M_α is a minimal (in fact, the least) such extension of $M_{\alpha-1}$. □

Proposition 31 *Let $\mathcal{M} \in \mathrm{Mod}_c^m(S)$ be a minimally stable coherent model of a sequent set $S \subseteq \mathrm{Seq}_2'$, then $M = D_{\mathcal{M}}$ is an answer set of the corresponding extended logic program Π_S.*

Proof: Let $\mathrm{Mod}_c^m((\Pi_S)^M) = \{\mathcal{M}'\}$. We have to show that $\mathcal{M}' = \mathcal{M}$. Denoting $M' = D_{\mathcal{M}'}$, we first prove that $M' \subseteq M$. Let $l \in M'$, i.e. there is $(l \leftarrow B') \in \Pi^M$, such that $\mathcal{M} \models B'$. Then there is a corresponding rule $(l \leftarrow B) \in [\Pi_S]$, such that $B' = B^+$, and $B^- \cap M = \emptyset$, and consequently $\mathcal{M} \models B$, implying that $l \in M$.

Assume that $\mathcal{M}$ is generated by $\mathcal{M}_0 \leq \ldots \leq \mathcal{M}_\kappa$. We show by induction on α that $M_\alpha \subseteq M'$ for $\alpha \leq \kappa$. For $\alpha = 0$, we have $M_0 = \emptyset \subseteq M'$. For a sucessor ordinal $\alpha = \beta + 1$, let $l \in M_{\beta+1} - M_\beta$. This means that $l \in \{k : (A \Rightarrow k) \in [S]\ \&\ [\mathcal{M}_{\beta-1}, \mathcal{M}] \models A\}$. Consequently, there is some rule $(l \leftarrow B) \in [\Pi_S]$, such that $[\mathcal{M}_\beta, \mathcal{M}] \models B$, implying that $(l \leftarrow B^+) \in (\Pi_S)^M$. Since by the induction hypthesis $M_\beta \subseteq M'$, it follows that $\mathcal{M}' \models B^+$, and consequently, $l \in M'$.

Finally, let $\alpha = \lambda$ be a limes ordinal. Then $M_\lambda = \bigcup_{\beta<\lambda} M_\beta \subseteq M'$, since by the induction hypothesis for all $\beta < \lambda$, $M_\beta \subseteq M'$. □

[18] There have been many meta-logical (notably modal logic) proposals, though.

Observation 19 *Since an ELP* Π *may have several minimal models, it holds that in general* $\mathrm{Mod}_*^m(\Pi) \neq \mathrm{Mod}_*^{pm}(\Pi)$. *However,*

$$\mathrm{Mod}_*^{ms}(\Pi) = \mathrm{Mod}_*^{pms}(\Pi)$$

Proof: There is exactly one minimal extension of $\mathcal{M}_{\alpha-1}$ satisfying all heads of sequents from $S_{[\mathcal{M}_{\alpha-1},\mathcal{M}]}$, namely $M_\alpha = \{l \in \mathrm{Lit}_0 : (l \leftarrow B) \in S_{[\mathcal{M}_{\alpha-1},\mathcal{M}]}\}$. $\Box$

6 Conclusion

Partial model theory, being a natural generalization of classical model theory, is able to capture many important distinctions arising in knowledge-based reasoning, such as explicit falsity vs. non-truth, or exact vs. inexact predicates. At the object level, these distinctions can be expressed by means of the two negations of partial logic. While the strong negation is useful to express the explicit falsity or incompatibility of some piece of information, the weak negation, as a non-persistent functor, can be used to express local Closed-World Assumptions and default rules.

We have shown in this paper how the fundamental notions of minimal, paraminimal and stable models in partial logic can be used to define the semantics of knowledge bases including relational and deductive databases, and extended logic programs.

References

[Acz68] P. Aczel. Saturated intuitionistic theories. In H. Schmidt, K. Schütte, and H. Thiele, editors, *Contributions to Mathematical Logic*, pages 1–13. North Holland, Amsterdam, 1968.

[Ben84] J.F.A.K. van Benthem. *A Manual of Intensional Logic*, volume 1 of *CSLI Lecture Notes*. CSLI publications, Stanford, CA, 1984.

[Bla86] S. Blamey. Partial logic. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Vol III: Alternatives to Classical Logic*, pages 1–70. Reidel, Dordrecht, 1986.

[DH94] J. Dietrich and H. Herre. Outline of nonmonotonic model theory. Technical Report NTZ-5:94, Universitaet Leipzig Institut fuer Informatik, 1994.

[Die94] J. Dietrich. Deductive basis of nonmonotonic inference operations. Technical Report NTZ-7,94, Universitaet Leipzig Institut fuer Informatik, Leipzig, 1994.

[Fef84] S. Feferman. Towards useful type free theories. *Journal of Symbolic Logic*, 49:75–111, 1984.

[FL94] M. Freund and D. Lehmann. Nonmonotonic inference operations: from finitary relations to infinitary operations. *Studia Logica*, 53:161–201, 1994.

[FLV92] J.E. Fenstad, T. Langholm, and E. Vespren. Representations and interpretations. In M. Rosner and R. Johnson, editors, *Computational Linguistics and Formal Semantics*, Studies in Natural Language Processing, pages 31–95. Cambridge University Press, Cambridge, 1992.

[Gil74] P.C. Gilmore. The consistency of partial set theory without extensionality. In *Proceedings of Symposia in Pure Mathematics*, volume Vol. 13 Part II. American Mathematical Society, Providence (RI), 1974.

[GL88] M. Gelfond and V. Lifschitz. The stable model semantics for logic programming. In *Proc. of Int. Conf. on Logic Programming.* MIT Press, 1988.

[GL90] M. Gelfond and V. Lifschitz. Logic programs with classical negation. In *Proc. of Int. Conf. on Logic Programming.* MIT Press, 1990.

[Her95] H. Herre. Generalized compactness of nonmonotonic inference operations. *Journal of Appl. Non-Classical Logics*, 5:121–135, 1995.

[Jas94] J.O.M. Jaspars. *Calculi for Constructive Communication: A Study of the Dynamics of Partial States.* PhD thesis, University of Tilburg, Amsterdam–Tilburg, June 1994. ILLC dissertation series 1994-4, and also ITK dissertation series 1994-1.

[Jas95] J.O.M. Jaspars. Partial up and down logic. *Notre Dame Journal of Formal Logic*, 36(1), 1995. to appear.

[Koe66] S. Koerner. *Experience and Theory.* Kegan Paul, London, 1966.

[Lan88] T. Langholm. *Partiality, Truth and Persistence*, volume 15 of *CSLI Lecture Notes.* CSLI publications, Stanford, CA, 1988.

[Lan91] T. Langholm. What is a horn clause in partial logic? In J. van Eijck, editor, *Logics in AI (JELIA'90)*, volume 478 of *Lecture Notes in Artificial Intelligence*, pages 65–77, Heidelberg, 1991. Springer-Verlag.

[Mak93] D. Makinson. General patterns in nonmonotonic reasoning. In D. Gabbay, editor, *Handbook of Logic in Artificial Intelligence and Logic Programming.* Oxford University Press, 1993.

[Mus95] R.A. Muskens. *Meaning and Partiality*, volume No.2 of *Studies in Logic, Language and Information.* CSLI Publications, Stanford (CA), 1995. To appear.

[Pri89] G. Priest. Reasoning about truth. *Artificial Intelligence*, 39:231–244, 1989.

[PW90] M. Papalaskari and S. Weinstein. Minimal consequence in sentential logic. *Journal of Logic Programming*, 9:19–31, 1990.

[TEG93] G. Gottlob T. Eiter and Y. Gurevich. Curbe your theory ! a circumscriptive approach for inclusive interpretation of dijunctive information. In *Proceedings of Int. Joint Conf. on Artificial Intelligence*, 1993.

[Thi92] E.G.C. Thijsse. *Partial Logic and Knowledge Representation.* PhD thesis, University of Tilburg, Tilburg, July 1992.

[Tho69] R.H. Thomason. A semantical study of constructible falsity. *Zeitschrift für Mathematischen Logik und Grundlagen der Mathematik*, 15:247–257, 1969.

[vG88] A. van Gelder. Negation as failure using tight derivations for general logic programs. In *Foundations of Deductive Databases and Logic Programming*, pages 149–176. Morgan Kaufmann, San Mateo (CA), 1988.

[Wag91] G. Wagner. A database needs two kinds of negation. In B. Thalheim and H.-D. Gerhardt, editors, *Proc. of the 3rd. Symp. on Mathematical Fundamentals of Database and Knowledge Base Systems*, Lecture Notes in Computer Science, pages 357–371. Springer-Verlag, 1991.

[Wag94a] G. Wagner. Transforming deductive into active databases. In N. Fuchs and G. Gottlob, editors, *Proc. of the 10th Workshop on Logic Programming.* Univ. Zürich, 1994.

[Wag94b] G. Wagner. *Vivid Logic – Knowledge-Based Reasoning with Two Kinds of Negation*, volume 764 of *Lecture Notes in Computer Science.* Springer-Verlag, Heidelberg, 1994.