Stable Models of Uncertain Logic Programs

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Abstract

Similar to normal logic programs (or deductive databases) which extend relational databases by allowing for deduction rules, uncertain logic programs (or deductive uncertain databases) result from the addition of deduction rules to uncertain databases. We show that in the same way as the minimal model semantics of relational databases can be refined to the stable model semantics for normal logic programs (see [GL88]), the minimal uncertain model semantics of uncertain databases can be refined to the stable uncertain model semantics for uncertain logic programs. Our notion of uncertainty in logic programs is based on the possibilistic logic of [DLP94].

Key words: stable models, uncertainty, possibilistic logic.

1 Introduction

As opposed to the traditional logical notion of a *theory* being a (possibly deductively closed) set of formulas, the emerging concept of a *knowledge base (KB)* is richer both in terms of the structure of a KB and in terms of the meaning-generating restrictions imposed upon it. Typically, a KB consists of *facts* and various kinds of *rules*. In this paper, we shall only consider *deduction rules*. Facts correspond to sentences of a suitably restricted language, and deduction rules correspond to non-schematic (Gentzen) sequents. While facts express extensional knowledge, rules express intensional knowledge. A set of facts and deduction rules is also called a *logic program*.

In real world knowledge bases like, for instance, relational or deductive databases, it is essential to be able to infer negative information by means of *minimal* (or *stable*) entailment, i.e. drawing inferences on the basis of minimal (or stable)

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models. Relational databases, being finite sets of tables the rows of which represent atomic sentences, have traditionally been viewed as finite models. On this account, answering a query F is rather based on the model relation, $\mathcal{M}_\Delta \models F$, where $\mathcal{M}_\Delta$ is the finite interpretation corresponding to the database Δ, and not on an inference relation. However, especially with respect to the generalization of relational databases (e.g. in order to allow for incomplete information), it seems to be more adequate to regard a relational database as a set of atomic sentences A_Δ, and to infer a query F whenever it holds in the unique minimal Herbrand model of A_Δ, i.e.

$$A_\Delta \vdash F :\Leftrightarrow \mathrm{Min}(\mathrm{Mod}_H(A_\Delta)) \subseteq \mathrm{Mod}(F) \Leftrightarrow \mathcal{M}_\Delta \models F$$

It turns out that for uncertain logic programs, like for normal logic programs, minimal models are not adequate because they are not able to account for the directedness of rules. Therefore, the more refined preference criterion of stability is needed to capture the class of intended models.

The structure of the paper is as follows. In section 3, we introduce the novel concept of an *uncertain Herbrand interpretation* which is closely related to the *possibility theory* of Zadeh, Dubois and Prade.[1] We show that this concept is a conservative extension of the concept of classical Herbrand interpretations. In section 4, we define a natural information ordering between uncertain interpretations which leads to the notions of minimal and stable models. Uncertain logic programs and stabe models are defined in section 5. Finally, we show in section 6 that normal logic programs under the stable models semantics of [GL88] can be faithfully embedded in uncertain logic programs under our stable semantics.

2 Preliminaries

A *signature* $\sigma = \langle Rel, Const, Fun \rangle$ consists of a set of relation symbols, a set of constant symbols, and a set of function symbols. U_σ denotes the set of all ground terms of σ. For a term tuple $t_1, \ldots, t_n$ we will also write $\vec{t}$ when its length is of no relevance.

The logical functors are $\neg, \wedge, \vee, |, \supset, \forall, \exists$; where $|$ and $\supset$ are called *exclusive* disjunction, and *material implication*, respectively. $L(\sigma)$ is the smallest set containing the atomic formulas of σ, and being closed with respect to the following conditions: if $F, G \in L(\sigma)$, then $\{\neg F,\ F \wedge G,\ F \vee G,\ F|G,\ F \supset G,\ \exists x F,\ \forall x F\} \subseteq L(\sigma)$.

$L^0(\sigma)$ denotes the corresponding set of sentences (closed formulas). For sublanguages of $L(\sigma)$ formed by means of a subset $\mathcal{F}$ of the logical functors, we

[1] See [Zad79, DP80, DLP94].

write $L(\sigma; \mathcal{F})$. With respect to a signature σ we define the following sublanguages: $\mathrm{At}(\sigma) = L(\sigma; \emptyset)$, the set of all atomic formulas (also called *atoms*); $\mathrm{Lit}(\sigma) = L(\sigma; \neg)$, the set of all *literals*. We introduce the following conventions. When $L \subseteq L(\sigma)$ is some sublanguage, L^0 denotes the corresponding set of sentences. If the signature σ does not matter, we omit it and write, e.g., L instead of $L(\sigma)$.

If Y is an partially ordered set, then $\mathrm{Min}(Y)$ denotes the set of all minimal elements of Y, i.e. $\mathrm{Min}(Y) = \{X \in Y \mid \neg\exists X' \in Y : X' < X\}$.

3 Model Theory

Definition 1 (Certainty Scale) *A certainty scale $\langle \boldsymbol{C}, 0, 1\rangle$ is a linearly ordered set $\boldsymbol{C}$ with least and greatest elements* 0 *and* 1.

Examples of certainty scales are the rational unit interval, or any discrete ordering of linguistic uncertainty values such as $\langle 0, ll, ql, vl, 1\rangle$, where ll stands for *little likely*, ql for *quite likely*, and vl for *very likely*. In the sequel, we assume that there is a fixed certainty scale $\boldsymbol{C}$ for which we simply write $[0, 1]$. If we want to exclude the value for complete uncertainty, we write $(0, 1] = \{v \in \boldsymbol{C} \mid v > 0\}$.

An *uncertain Herbrand interpretation* of the language $L(\sigma)$ is based on uncertain relations over the Herbrand universe U_σ.

Definition 2 (Uncertain Interpretation) *Let $\sigma = \langle Rel, Const, Fun\rangle$ be a signature. An uncertain Herbrand σ-interpretation $\mathcal{I}$ over a certainty scale $\boldsymbol{C}$ consists of:*

1. *A set $U_\mathcal{I}$, called the* universe *or* domain *of $\mathcal{I}$ which is equal to the set of all ground terms, $U_\mathcal{I} = U_\sigma$;*
2. *an assignment $c^\mathcal{I} = c$ to every constant symbol $c \in Const$;*
3. *an assignment of a function $f^\mathcal{I} : U_\sigma^{a(f)} \to U_\sigma$ to every function symbol $f \in Fun$, where $a(f)$ denotes the arity of f, such that $f^\mathcal{I}(\vec{t}) = f(\vec{t})$;*
4. *an assignment of a function $r^\mathcal{I} : U_\sigma^{a(r)} \to \boldsymbol{C}$ to every relation symbol $r \in Rel$, where $a(r)$ denotes the arity of r; the function $r^\mathcal{I}$ is also called an* uncertain relation.

The class of all uncertain Herbrand σ-interpretations is denoted by $\boldsymbol{I}_H(\sigma)$. In the sequel we shall simply say 'interpretation' instead of 'uncertain Herbrand interpretation'. It is important to note that an interpretation assigns certainty values to atomic sentences only (strictly speaking, it is only an interpretation of the lanuage $\mathrm{At}(\sigma)$, therefore). In order to interpret also complex sentences, an interpretation has to be complemented with a *certainty valuation*.

Definition 3 (Certainty Valuation) *A function $C : L^0(\sigma) \to [0,1]$ is called a certainty valuation, if it satisfies the following conditions:*

(0) $C(F \wedge \neg F) = 0$.

($\neg a$) $C(\neg a) = 1$ *if* $C(a) = 0$, *for all* $a \in \mathrm{At}^0(\sigma)$.

($\wedge$) $C(F \wedge G) = \min(C(F), C(G))$.

($\vee$) $C(F \vee G) \geq \max(C(F), C(G))$.

($\exists$) $C(\exists x F(x)) \geq \max\{C(F(t)) \mid t \in U_\sigma\}$.

($\forall$) $C(\forall x F(x)) = \min\{C(F(t)) \mid t \in U_\sigma\}$.

and, in addition, respects the following rewrite rules:

$$
\begin{array}{rcl rcl}
\neg\neg F & \longrightarrow & F & & & \\
\neg(F \wedge G) & \longrightarrow & \neg F \vee \neg G & \neg(F \vee G) & \longrightarrow & \neg F \wedge \neg G \\
\neg\exists x F & \longrightarrow & \forall x \neg F & \neg\forall x F & \longrightarrow & \exists x \neg F \\
F \supset G & \longrightarrow & \neg F \vee G & F|G & \longrightarrow & (F \wedge \neg G) \vee (G \wedge \neg F)
\end{array}
$$

in the sense that for every rewrite rule $LHS \longrightarrow RHS$, it holds that $C(LHS) = C(RHS)$.

Our certainty valuations are closely related to the *necessity measures* of possibilistic logic (see, e.g., [DLP94]). They satisfy the additional condition ($\neg a$), however, which is essential for the embeddability of classical models (see claim 1). Like necessity measures, they are not truth-functional. Consequently, uncertain model theory, as outlined in the present paper, does not yield a multiple-valued logic.

The following observation is a straightforward consequence of condition (0).

Observation 1 *Let $F \in L^0(\sigma)$, and let C be a certainty valuation. Then, $C(F) = 0$ or $C(\neg F) = 0$.*

Observation 2 *For positive sentences $F \in L(\sigma; \wedge, \vee, \exists, \forall)$, it holds that $C(F) = 0$ implies $C(\neg F) = 1$.*

Proof: by straightforward induction on F.

Notice that for an atom a, $C(\neg a) = 0$ does not imply $C(\neg\neg a) = 1$ since this would entail that $C(a) = 1$, but it is perfectly possible that $C(\neg a) = 0$ and $0 < C(a) < 1$.

Definition 4 ($\mathcal{I}$-Compatible Certainty Valuation) *Let $\mathcal{I} \in \boldsymbol{I}_H(\sigma)$. A certainty valuation $C : L^0(\sigma) \to [0,1]$ is called an $\mathcal{I}$-compatible certainty valuation, symbolically $C \in \mathrm{Cert}(\mathcal{I})$, if it satisfies the following condition:*

$$(a) \quad C(r(t_1, \ldots, t_n)) = r^{\mathcal{I}}(t_1, \ldots, t_n)$$

An *uncertain model* consists of an uncertain interpretation and a certainty valuation compatible with it.

Definition 5 (Model Relation) *The model relation* $\models$ *between an interpretation* $\mathcal{I} \in \boldsymbol{I}_H(\sigma)$*, a certainty valuation* $C \in \mathrm{Cert}(\mathcal{I})$*, and a valuated sentence* $F{:}\nu \in L^0(\sigma) \times [0,1]$*, is defined as follows:*

$$\mathcal{I}, C \models F{:}\nu \quad \textit{iff} \quad C(F) \geq \nu$$

An *instantiation* over an interpretation $\mathcal{I}$ is a function $\iota : \mathit{Var} \to U_\sigma$, assigning ground terms to individual variables, which can be naturally extended to arbitrary terms by

$$\iota(f(t_1, \ldots, t_n)) = f^{\mathcal{I}}(\iota(t_1), \ldots, \iota(t_n))$$

Note that for a constant c, being a 0-ary function, we have $\iota(c) = c^{\mathcal{I}} = c$. An instantiation ι is also a function $\iota : L(\sigma) \to L^0(\sigma)$, assiging a sentence $F\iota$ to a formula F.

The model relation for a valuated formula $F{:}\nu \in L(\sigma) \times [0,1]$ is then defined by

$$\mathcal{I}, C \models F{:}\nu \quad \text{iff} \quad \mathcal{I}, C \models F\iota{:}\nu \text{ for all instantiations } \iota : \mathit{Var} \to U_\sigma$$

Let $X \subseteq L^0(\sigma) \times [0,1]$ be a set of valuated sentences. The class of all models of X is defined by

$$\mathrm{Mod}_H(X) = \{\langle \mathcal{I}, C\rangle \mid \mathcal{I} \in \boldsymbol{I}_H(\sigma) \;\;\&\;\; C \in \mathrm{Cert}(\mathcal{I}) \;\;\&\;\; \mathcal{I}, C \models F \;\text{ f.a. }\; F \in X\}$$

and $\models_H$ denotes the corresponding consequence relation, i.e. $X \models_H F$ iff $\mathrm{Mod}_H(X) \subseteq \mathrm{Mod}_H(F)$.

Example 1 $\{p(b){:}0.7,\ q(d,c){:}0.2,\ q(d,b){:}0.4\} \models_H [p(b) \wedge \exists x q(d,x)] : 0.3$

Definition 6 (Diagram) *The* diagram *of an uncertain interpretation* $\mathcal{I} \in \boldsymbol{I}_H(\sigma)$ *is defined as*

$$D_{\mathcal{I}} = \{r(t_1, \ldots, t_n){:}\nu \in \mathrm{At}^0(\sigma) \times [0,1] \mid \nu = r^{\mathcal{I}}(t_1, \ldots, t_n) > 0\}$$

Notice that in the dagram of an uncertain interpretation only positive information is represented, and uninformative sentences of the form $a{:}0$ are not included.

Definition 7 (Normalization) *For a collection of sets of valuated atoms* $\mathcal{A} \subseteq 2^{\mathrm{At}^0(\sigma)\times[0,1]}$*, we define its normalization* $\mathrm{Norm}(\mathcal{A}) = \{A \in \mathcal{A} : A = \mathrm{Max}(A)\}$*, where* $\mathrm{Max}(A) = \{a{:}\nu \in A \mid \nu \geq \mu \;\; \textit{f.a.} \;\; a{:}\mu \in A\}$

Observation 3 *Uncertain Herbrand interpretations can be identified with their diagrams. In other words, there is a one-to-one correspondence between* $\boldsymbol{I}_H(\sigma)$ *and* $\mathrm{Norm}(2^{\mathrm{At}^0(\sigma)\times(0,1]})$.

Consequently, uncertain Herbrand interpretations over σ can be considered as normalized subsets of $\mathrm{At}^0(\sigma) \times (0,1]$. In the sequel, we shall identify an interpretation with its diagram whenever it is convenient.

Claim 1 *An ordinary Herbrand interpretation $I \subseteq \mathrm{At}^0(\sigma)$ can be embedded in an uncertain Herbrand interpretation $I^u = \{a{:}1 \mid a \in I\}$, such that for all $F \in L^0(\sigma)$,*

$$I \models F \;\Rightarrow\; I^u, C \models F{:}1 \;\; \textit{f.a.} \;\; C \in \mathrm{Cert}(I^u)$$

Proof (by induction on sentences): For atoms, the assertion follows immediately from the definition of I^u. For $\wedge$ and $\vee$ the induction argument is straightforward, likewise for $\forall$ and $\exists$. We only treat the case of negation: $I \models \neg a$ iff $a \notin I$, implying that $a{:}\nu \notin I^u$, and consequently for all $C \in \mathrm{Cert}(I^u)$, it holds that $C(a) = 0$, hence $C(\neg a) = 1$, and $I^u, C \models \neg a : 1$. The remaining cases of negated complex sentences are handled by the DeMorgan-style rewrite rules. □

4 Databases and Minimal Models

Definition 8 (Uncertain Database) *An uncertain database is a finite set of finite uncertain relations (uncertain tables), corresponding to a normalized set of certainty-valuated atoms.*

Obviously, an uncertain database corresponds to a finite uncertain interpretation.

Example 2 *The uncertain database consisting of the two tables*

$$P = \begin{array}{|c|c|}\hline d & 1 \\ b & 0.7 \\ c & 0.1 \\ \hline\end{array} \qquad Q = \begin{array}{|cc|c|}\hline d & c & 0.8 \\ d & b & 0.3 \\ \hline\end{array}$$

corresponds to $X_2 = \{p(d){:}1,\ p(b){:}0.7,\ p(c){:}0.1,\ q(d,c){:}0.8,\ q(d,b){:}0.3\}$.

Definition 9 (Informational Extension) *Let $\mathcal{I}, \mathcal{I}' \in \boldsymbol{I}_H$ be two interpretations. We say that $\mathcal{I}'$* informationally extends *$\mathcal{I}$, or $\mathcal{I}'$ is* at least as informative as *$\mathcal{I}$, symbolically $\mathcal{I} \leq \mathcal{I}'$, if f.a. $a{:}\nu \in D_{\mathcal{I}}$ exists $a{:}\mu \in D_{\mathcal{I}'}$, such that $\nu \leq \mu$.*

Observation 4 *$\langle \boldsymbol{I}_H(\sigma), \leq, \emptyset \rangle$ is a complete lattice with least element $\emptyset$.*

The partial ordering of atomic interpretations can be naturally extended to full interpretations by defining

$$\langle \mathcal{I}, C \rangle \leq \langle \mathcal{I}', C' \rangle \;:\Longleftrightarrow\; \mathcal{I} \leq \mathcal{I}' \;\&\; C \leq C'$$

Observation 5 (Minimal Certainty Valuation) *For every interpretation $\mathcal{I} \in \boldsymbol{I}_H$, there is a unique minimal certainty valuation $C_{\mathcal{I}} \in \mathrm{Cert}(\mathcal{I})$, which satisfies*

$$\begin{aligned} C_{\mathcal{I}}(F \vee G) &= \max(C_{\mathcal{I}}(F), C_{\mathcal{I}}(G)) \\ C_{\mathcal{I}}(\exists x F(x)) &= \max\{C_{\mathcal{I}}(F(t)) \mid t \in U_\sigma\} \end{aligned}$$

Definition 10 (Minimal Model) *For $F \in L^0(\sigma) \times [0,1] \supseteq X$, we define* $\mathrm{Mod}_m(X) = \mathrm{Min}(\mathrm{Mod}_H(X))$*, and minimal entailment: $X \models_m F$ iff* $\mathrm{Mod}_m(X) \subseteq \mathrm{Mod}_H(F)$.

Notice that if $\langle \mathcal{M}, C \rangle$ is a minimal model, then $C = C_{\mathcal{M}}$. In the sequel, we shall also simply write $\mathcal{M} \models F$ instead of $\mathcal{M}, C_{\mathcal{M}} \models F$.

Observation 6 *A valuated sentence F can be inferred from an uncertain database X if it is minimally entailed: $X \models_m F$.*

We distinguish between two kinds of satisfaction for open formulas on the basis of a model: a valuated formula is satisfied by ordinary answers (i.e. substitions), while an ordinary formula is satisfied by valuated answers.

Definition 11 (Answers) *Let $\mathcal{I} \in \boldsymbol{I}_H(\sigma)$, $C \in \mathrm{Cert}(\mathcal{I})$, and $\nu \in [0,1]$. Then, for a sentence $F \in L^0(\sigma)$,*

$$\mathsf{Ans}_{\mathcal{I},C}(F) = C(F)$$

while for a formula $F \in L(\sigma) - L^0(\sigma)$,

$$\begin{aligned} \mathsf{Ans}_{\mathcal{I},C}(F{:}\nu) &= \{\iota \in {U_\sigma}^{\mathit{Var}} \mid \mathcal{I}, C \models F\iota : \nu\} \\ \mathsf{Ans}_{\mathcal{I},C}(F) &= \{\iota{:}\nu \in {U_\sigma}^{\mathit{Var}} \times [0,1] \mid \nu = C(F\iota)\} \end{aligned}$$

Instead of $\mathsf{Ans}_{\mathcal{I},C_{\mathcal{I}}}$ we simply write $\mathsf{Ans}_{\mathcal{I}}$. Whenever convenient, we use the substitution notation $\iota^{x_1,\ldots,x_n}_{c_1,\ldots,c_n}$ to represent the set of instantiations $\{\iota \in {U_\sigma}^{\mathit{Var}} \mid \iota(x_i) = c_i \text{ for } i = 1, \ldots, n\}$.

Definition 12 (Less Informative Answer Set) *We say that an answer set A' is at least as informative as an answer set A, symbolically $A \leq A'$, if*

1. *$A, A' \subseteq {U_\sigma}^{\mathit{Var}}$, and $A \subseteq A'$.*
2. *$A, A' \subseteq {U_\sigma}^{\mathit{Var}} \times [0,1]$, and f.a. $\iota{:}\nu \in A$ exists $\iota{:}\mu \in A'$ s.th. $\nu \leq \mu$.*

Example 3 *Viewing the uncertain database X_2 from example 2 as the diagram of an interpretation $\mathcal{I}_2$, we get*

$$\begin{aligned} \mathsf{Ans}_{\mathcal{I}_2}(q(x,y) \wedge p(y) : 0.2) &= \iota^{x,y}_{d,b} \\ \mathsf{Ans}_{\mathcal{I}_2}(q(x,y) \wedge p(y)) &= \iota^{x,y}_{d,c} : 0.1 \cup \iota^{x,y}_{d,b} : 0.3 \end{aligned}$$

Observation 7 *An interpretation represents the unique minimal model of its diagram in the following sense: f.a.* $\mathcal{I} \in \boldsymbol{I}_H$, $\mathrm{Mod}_m(D_{\mathcal{I}}) = \{\langle \mathcal{I}, C_{\mathcal{I}} \rangle\}$, *and consequently, a valuated sentence* $F \in L^0 \times [0,1]$ *is satisfied by an interpretation iff it is minimally entailed by its diagram:* $\mathcal{I}, C_{\mathcal{I}} \models F$ *iff* $D_{\mathcal{I}} \models_m F$.

Definition 13 (Persistent Formula) *A formula* $F \in L(\sigma)$ *is called* persistent *if for arbitrary interpretations* $\mathcal{I}_1, \mathcal{I}_2 \in \boldsymbol{I}_H(\sigma)$, $C_i \in \mathrm{Cert}(\mathcal{I}_i)$, *with* $\langle \mathcal{I}_1, C_1 \rangle \leq \langle \mathcal{I}_2, C_2 \rangle$, *for every instantiation* $\iota : Var \to U_\sigma$, *and for every certainty value* $\nu \in [0,1]$, *it holds that* $\mathcal{I}_1, C_1 \models F\iota{:}\nu$ *implies* $\mathcal{I}_2, C_2 \models F\iota{:}\nu$.

Observation 8 *Every formula* $F \in L(\sigma; \wedge, \vee, \exists, \forall)$ *is persistent.*

5 Sequents and Stable Models

Traditionally, Gentzen sequents are used in a schematic way in sequent calculi in order to express valid transitions from one argument schema to another. In other words, a sequent in a sequential inference rule stands for a whole class of propositional substitution instances. Here, we propose to use sequents in a non-schematic way for the purpose of representing rule knowledge. A sequent, then, is not a schematic but a concrete expression representing a piece of rule knowledge.

Definition 14 (Sequent) *A sequent s is an expression of the form*

$$F_1, \ldots, F_m \;\Rightarrow\; G_1, \ldots, G_n$$

where either $F_i, G_j \in L(\sigma)$, *or* $F_i, G_j \in L(\sigma) \times [0,1]$, *for* $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$. *The* body *of s, denoted by* Bs, *is given by* $\{F_1, \ldots, F_m\}$, *and the* head *of s, denoted by* Hs, *is given by* $\{G_1, \ldots, G_n\}$. $\mathrm{Seq}(\sigma, \boldsymbol{C})$ *denotes the class of all sequents s such that* $Hs, Bs \subseteq L(\sigma) \times \boldsymbol{C}$, *and* $\mathrm{Seq}(\sigma)$ *denotes the class of all sequents s such that* $Hs, Bs \subseteq L(\sigma)$.

Notice that we did not define uncertain sequents $B \Rightarrow H : \nu$ which would themselves be qualified with a certainty degree ν. The definition of satisfaction for such uncertain sequents requires a 'multiplication' operation $\nu\mu$ for certainty values which would exclude discrete (linguistic) certainty scales. It is a straightforward modification of our present work to make this restriction and consider only interpretations over the rational (or real-valued) unit interval $[0,1]$ which give also meaning to uncertain sequents.

Definition 15 (Model of a Sequent) *Let* $\mathcal{I} \in \boldsymbol{I}_H(\sigma)$, *and* $C \in \mathrm{Cert}(\mathcal{I})$, *and s be any sequent. Then,*

$$\mathcal{I}, C \models s \quad \textit{iff} \quad \inf_{F \in Bs} \mathsf{Ans}_{\mathcal{I},C}(F) \leq \sup_{G \in Hs} \mathsf{Ans}_{\mathcal{I},C}(G)$$

For instance, refering to our previous example,

$$\mathcal{I}_2 \models q(x,y) \wedge p(y) : 0.2,\ \neg q(y,x) : 1 \Rightarrow p(y){:}0.5$$

since $\mathsf{Ans}_{\mathcal{I}_2}(q(x,y) \wedge p(y) : 0.2,\ \neg q(y,x) : 1)$

$$\begin{aligned}
&= \mathsf{Ans}_{\mathcal{I}_2}(q(x,y) \wedge p(y) : 0.2) \cap \mathsf{Ans}_{\mathcal{I}_2}(\neg q(y,x) : 1)\\
&= \iota^{x,y}_{d,b} \cap (\iota^{x,y}_{d,b} \cup \iota^{x,y}_{b,b} \cup \iota^{x,y}_{c,b} \cup \ldots)\\
&= \iota^{x,y}_{d,b}\\
&\subseteq \mathsf{Ans}_{\mathcal{I}_2}(p(y){:}0.5)\\
&= \iota^{y}_{b} \cup \iota^{y}_{d}
\end{aligned}$$

Likewise, $\mathcal{I}_2 \models q(x,y) \Rightarrow p(x)$, since

$$\begin{aligned}
\mathsf{Ans}_{\mathcal{I}_2}(q(x,y)) &= \iota^{x,y}_{d,c} : 0.1 \cup \iota^{x,y}_{d,b} : 0.3\\
&\leq \mathsf{Ans}_{\mathcal{I}_2}(p(x))\\
&= \iota^{x}_{d} : 1 \cup \iota^{x}_{b} : 0.7 \cup \iota^{x}_{c} : 0.1
\end{aligned}$$

Finally, an *uncertain logic program* is defined as a finite set of sequents where in the case of facts we usually write just F instead of $\Rightarrow F$. For instance,

$$S = \begin{cases} p|q \\ p \vee \neg q \Rightarrow r \supset s,\ u \\ p \wedge q : 0.5,\ \neg r|s : 0.2 \Rightarrow \neg u : 1,\ v \wedge w : 0.5 \end{cases}$$

is an uncertain logic program. We define the following classes of sequents corresponding to positive, normal, and generalized uncertain logic programs, respectively.

$$\begin{aligned}
\mathrm{PULP}(\sigma) &= \{s \in \mathrm{Seq}(\sigma) \mid Hs \in \mathrm{At}(\sigma),\ Bs \subseteq \mathrm{At}(\sigma)\\
&\quad \cup \{s \in \mathrm{Seq}(\sigma, \boldsymbol{C}) \mid Hs \in \mathrm{At}(\sigma) \times [0,1],\ Bs \subseteq \mathrm{At}(\sigma) \times [0,1]\}\\
\mathrm{NULP}(\sigma) &= \{s \in \mathrm{Seq}(\sigma) \mid Hs \in \mathrm{At}(\sigma),\ Bs \subseteq \mathrm{Lit}(\sigma)\}\\
&\quad \cup \{s \in \mathrm{Seq}(\sigma, \boldsymbol{C}) \mid Hs \in \mathrm{At}(\sigma) \times [0,1],\ Bs \subseteq \mathrm{Lit}(\sigma) \times [0,1]\}\\
\mathrm{GULP}(\sigma) &= \{s \in \mathrm{Seq}(\sigma) \mid Hs \subseteq L(\sigma; \wedge, \vee),\ Bs \subseteq L(\sigma; \neg, \wedge, \vee, |, \supset)\}\\
&\quad \cup \{s \in \mathrm{Seq}(\sigma, \boldsymbol{C}) \mid Hs \subseteq L(\sigma; \wedge, \vee) \times [0,1],\\
&\qquad\qquad Bs \subseteq L(\sigma; \neg, \wedge, \vee, |, \supset) \times [0,1]\}
\end{aligned}$$

Instead of the sequent notation $B \Rightarrow H$, we also use the rule notation $\bigvee H \leftarrow \bigwedge B$. For $S \subseteq \mathrm{GULP}(\sigma)$, we define the model operators

$$\begin{aligned}
\mathrm{Mod}_H(S) &= \{\langle \mathcal{I}, C\rangle : \mathcal{I}, C \models s \ \text{ f.a. } \ s \in S\}\\
\mathrm{Mod}_m(S) &= \mathrm{Min}(\mathrm{Mod}_H(S))
\end{aligned}$$

The associated entailment relations are defined as follows:

$$S \models_* F \quad \text{iff} \quad \mathrm{Mod}_*(S) \subseteq \mathrm{Mod}_H(F)$$

where $* = H, m$, and $F \in L(\sigma) \times [0,1]$.

Observation 9 *Let $S \subseteq \mathrm{GULP}$ be a generalized logic program. Then, the models of S agree with the models of the Herbrand instantion $[S]$ of S:* $\mathrm{Mod}_H(S) = \mathrm{Mod}_H([S])$.

Let $\boldsymbol{K}$ be a class of models. We write $\boldsymbol{K} \models F$ iff $\mathcal{I}, C \models F$ for all $\langle \mathcal{I}, C \rangle \in \boldsymbol{K}$. We denote the set of all ground sequents from a sequent set S which are applicable in $\boldsymbol{K}$ by

$$S_K = \{s \in [S] : \boldsymbol{K} \models Bs\}$$

If $\boldsymbol{K}$ is a singleton, we omit brackets.

5.1 Persistent Rules and Minimal Models

A rule s is called *persistent*, if all body formulas $F \in Bs$ are persistent. For instance, all rules from PULP are persistent.

Observation 10 *For a program $S \subseteq \mathrm{PULP}$, where the rule heads consist of a single (valuated) atom, and the bodies are persistent, there is a unique minimal model, denoted $\mathcal{M}_S$, i.e.* $\mathrm{Mod}_m(S) = \{\langle \mathcal{M}_S, C_{\mathcal{M}_S} \rangle\}$.

Proof: We have to show that the interpretation $\mathcal{M}_S$ with the diagram

$$M_S = \{a{:}\nu \mid \nu > 0 \ \ \& \ \ \mathrm{Mod}_H(S) \models a{:}\nu\}$$

is a model of S. Obviously, if it is a model, it is the least one.

Let $(B \Rightarrow a{:}\nu) \in [S]$ and $\mathcal{M}_S \models B$. By the persistence of B we have $\mathcal{M} \models B$ for every model $\mathcal{M} \in \mathrm{Mod}_H(S)$. This implies that for every $\mathcal{M} \in \mathrm{Mod}_H(S)$, we have $\mathcal{M} \models a{:}\nu$, i.e. $a{:}\mu \in D_{\mathcal{M}}$ with $\nu \leq \mu$. Consequently, $a{:}\nu \in M_S$. $\Box$

Example 4 (Least Uncertain Path) [2] *In a fuzzy graph, weights $\nu \in [0, 1]$ are assigned to all edges, i.e. an edge $\langle a, b \rangle$ may be represented as an uncertain fact $e(a, b){:}\nu$. For instance, the facts $X =$*

$$\{\, e(a, b){:}0.5,\ e(a, c){:}0.7,\ e(b, d){:}0.6,\ e(d, b){:}0.6,\ e(c, d){:}0.8,\ e(b, e){:}0.9,\ e(d, e){:}0.2 \,\}$$

define a fuzzy graph. The weight of a path is the minimum of the weights of the edges composing it. This is implicitly expressed by the rules defining a path:

$$R = \begin{cases} p(x, y) \leftarrow e(x, y) \\ p(x, z) \leftarrow e(x, y) \wedge p(y, z) \end{cases}$$

The problem of finding the least uncertain path between the two nodes a and e can be solved by computing the least model $\mathcal{M}_S$ of the positive uncertain logic program $S = X \cup R$, and then determine

$$\mathsf{Ans}_{\mathcal{M}_S}(p(a, e)) = 0.6 = C_{\mathcal{M}_S}(e(a, c) \wedge e(c, d) \wedge e(d, b) \wedge e(b, e))$$

[2] This example is taken from [DLP91].

5.2 Non-Persistent Rules and Stable Models

When an uncertain logic program contains non-persistent rules, it may have minimal models which are not intended. This is illustrated by the following example.

Example 5
Let $S_5 = \{p(c){:}vl,\ r(x) \leftarrow p(x) \wedge \neg q(x)\}$ *be a normal uncertain logic program over the discrete certainty scale* $[0,1] = \langle 0, ll, ql, vl, 1\rangle$. *The minimal models of* S_5 *are*

$$\begin{array}{rcl} M_1 & = & \{p(c){:}vl,\ r(c){:}vl\} \\ M_2 & = & \{p(c){:}vl,\ q(c){:}ll\} \end{array}$$

Since we want to obtain the conclusion $r(c){:}vl$, *only* M_1 *is acceptable as an intended model. The fact* $q(c){:}ll$ *is not grounded in* S_5.

Therefore, instead of minimality we need a more refined preference criterion which allows to select the intended models of a set of sequents from its Herbrand models.

Definition 16 (Interpretation Interval) $[\mathcal{M}_1, \mathcal{M}_2] = \{\mathcal{M} \in \boldsymbol{I}_H : \mathcal{M}_1 \leq \mathcal{M} \leq \mathcal{M}_2\}$

The following definition of a *stable model* of an uncertain logic program is inspired by the definition of a *stable closure* of a set of rules in [Wag94], and almost identical to the definition of stable models for normal logic programs in [HW96].

Definition 17 (Stable Model) *A model* $\langle \mathcal{M}, C_\mathcal{M}\rangle \in \mathrm{Mod}_H(S)$ *is called a* stable model *of* $S \subseteq \mathrm{GULP}(\sigma)$, *symbolically* $\langle \mathcal{M}, C_\mathcal{M}\rangle \in \mathrm{Mod}_s(S)$, *if there is a chain of uncertain Herbrand interpretations* $\mathcal{I}_0 \leq \ldots \leq \mathcal{I}_\kappa$, *such that* $\mathcal{M} = \mathcal{I}_\kappa$, *and*

1. $\mathcal{I}_0 = \emptyset$.

2. *For successor ordinals* α *with* $0 < \alpha \leq \kappa$, $\mathcal{I}_\alpha$ *is a minimal extension of* $\mathcal{I}_{\alpha-1}$ *satisfying the heads of all sequents whose bodies hold in* $[\mathcal{I}_{\alpha-1}, \mathcal{M}]$, *i.e.*

$$\mathcal{I}_\alpha \in \mathrm{Min}\{\mathcal{I} \in \boldsymbol{I}_H(\sigma) \mid \mathcal{I} \geq \mathcal{I}_{\alpha-1},\ \textit{and}\ \mathcal{I} \models \bigvee Hs,\ \textit{f.a.}\ s \in S_{[\mathcal{I}_{\alpha-1}, \mathcal{M}]}\}$$

3. *For limit ordinals* $\lambda \leq \kappa$, $\mathcal{I}_\lambda = \sup_{\alpha<\lambda} \mathcal{I}_\alpha$.

We also say that $\mathcal{M}$ *is* generated *by the* S-stable chain $\mathcal{I}_0 \leq \ldots \leq \mathcal{I}_\kappa$.

Stable entailment is defined as follows:

$$S \models_s F \quad \text{iff} \quad \mathrm{Mod}_s(S) \subseteq \mathrm{Mod}_H(F)$$

where $F \in L(\sigma)$.

Example 5 (continued) *Only* M_1 *is a stable model of* S_5 *generated by the stable chain* $\emptyset \leq \{p(c){:}vl\} \leq \{p(c){:}vl, r(c){:}vl\}$. M_2 *is not a minimal extension of* $I_1 = \{p(c){:}vl\}$ *satisfying* $\bigvee Hs$ *for all* $s \in (S_5)_{[I_1,M_2]}$, *simply because* $(S_5)_{[I_1,M_2]} = \emptyset$, *i.e. the extension condition is trivially satisfied, and hence* I_1 *is the only minimal extension of* I_1. *Consequently,* $\mathrm{Mod}_s(S_5) = \{\mathcal{M}_1\}$, *and hence* $S_5 \models_s r(c){:}vl$.

Stable models do not exist in all cases. For instance, $S = \{p \leftarrow \neg p\}$ has exactly one minimal model, $\mathrm{Mod}_m(S) = \{\{p{:}ll\}\}$, which is not stable, however. We call a program without stable models *unstable.*

Observation 11 *Minimal and stable models coincide for positive uncertain logic programs, i.e. if* $S \subseteq \mathrm{PULP}$, *then* $\mathrm{Mod}_s(S) = \mathrm{Mod}_m(S)$.

6 Embedding Normal Logic Programs

For $B \subseteq \mathrm{Lit}(\sigma)$, let B^- denote the set of atoms which occur negated in B, i.e. $B^- := \{a \in \mathrm{At}(\sigma) : \neg a \in B\}$, and let $B^+ = \{a \in \mathrm{At}(\sigma) : a \in B\}$. It holds that for any $B \subseteq \mathrm{Lit}^0(\sigma)$, and any ordinary Herbrand interpretation $I \subseteq \mathrm{At}^0(\sigma)$,

$$I \models B \quad \text{iff} \quad B^+ \subseteq I \ \& \ B^- \cap I = \emptyset$$

Definition 18 (Immediate Consequence Operator) *Let* Π *be a normal logic program, and* $I \subseteq \mathrm{At}$. *Then,*

$$T_\Pi(I) = \{a \in \mathrm{At}^0 : \exists (a \leftarrow B) \in [\Pi], \textit{ s.th. } I \models B\}$$

is called the immediate consequence operator *associated with* Π.

Definition 19 (Gelfond-Lifschitz 1988) *Let* Π *be a normal logic program, and* $I \subseteq \mathrm{At}$. *Then the* Gelfond-Lifschitz transformation *of* Π *with respect to* I *is defined as*

$$\Pi^I = \{a \leftarrow B^+ \mid (a \leftarrow B) \in [\Pi], \textit{ and } B^- \cap I = \emptyset\}$$

and the Gelfond-Lifschitz operator Γ_Π *is defined as follows:* $\Gamma_\Pi(I)$ *denotes the unique minimal model of* Π^I, *i.e.* $\Gamma_\Pi(I) = \mathcal{M}_{\Pi^I}$. *According to Gelfond and Lifschitz, an atom* a *follows from* Π, *symbolically* $\Pi \vdash a$, *if it is in all fixpoints of* Γ_Π.

We shall show now that a normal logic program Π can be faithfully embedded in an uncertain logic program

$$\Pi^u = \{a{:}1 \leftarrow B^u \mid (a \leftarrow B) \in \Pi\}$$

where $B = l_1 \wedge \ldots \wedge l_m$, $l_i \in \mathrm{Lit}$, and $B^u = l_1{:}1, \ldots, l_m{:}1$.

Claim 2 *Uncertain logic programs are a conservative extension of normal logic programs, i.e.* $\Pi^u \models_s a{:}1$ *whenever* $\Pi \vdash a$.

Proof sketch: We show that every fixpoint M of Γ_Π corresponds to a stable uncertain model $M^u = \{a{:}1 \mid a \in M\}$, such that

$$\mathrm{Mod}_s(\Pi^u) = \{M^u \in \boldsymbol{I}_H \mid \Gamma_\Pi(M) = M\}$$

($\supseteq$) Let M be a fixpoint of Γ_Π, i.e. $\mathrm{Mod}_m(\Pi^M) = \{M\}$. Using the immediate consequence operator associated with Π^M we construct the sequence $\{I_n\}_{n<\omega}$, such that $I_0 = \emptyset$, $I_{n+1} = T_{\Pi^M}(I_n)$, and $I_\omega = \bigcup_{n<\omega} I_n$. It is well-known that $I_\omega = M$. We claim that $\{I_n^u\}_{n<\omega}$ is a Π^u-stable chain generating M^u, i.e. I_{n+1}^u is a minimal extension of I_n^u such that for all rules $a{:}1 \leftarrow B^u \in \Pi^u$ with $[I_n^u, M^u] \models B^u$, we have $I_{n+1}^u \models a{:}1$. Clearly, by construction, $I_n^u \leq I_{n+1}^u \leq M^u$. It follows from $a{:}1 \leftarrow B^u \in \Pi^u$ and $[I_n^u, M^u] \models B^u$, that $I_n \models B^+$ and $M \cap B^- = \emptyset$, implying $a \leftarrow B^+ \in \Pi^M$, and $a \in T_{\Pi^M}(I_n) = I_{n+1}$. Consequently, $I_{n+1}^u \models a{:}1$. It is easy to see that I_{n+1}^u is a minimal extension of I_n^u with these properties.

($\subseteq$) Clearly, every stable model of Π^u is of the form $M^u = \{a{:}1 \mid a \in \mathrm{At}\}$. Using the categorical projection $M = \{a \in \mathrm{At} \mid a{:}1 \in M^u\}$ of M^u and the immediate consequence operation associated with Π^M, we consider again the sequence generated by the iteration of T_{Π^M}: $I_0 = \emptyset, I_{n+1} = T_{\Pi^M}(I_n)$, and $I_\omega = \bigcup_{n<\omega} I_n$. It can be shown now by induction that $M = I_\omega$. $\Box$

Any reasonable concept of an uncertain logic program has to be a conservative extension of normal logic programs. The faithful embeddablity of normal logic programs, therefore, seems to be a knock-out criterion for any new proposal.

7 Related Work

Early attempts to introduce uncertainty into logic pogramming are [Sha83] and [vEm86]. Both proposals, however, lack a model-theoretic foundation and are, like the treatment of 'certainty factors' in MYCIN, somewhat ad hoc. There are several proposals, based on the fuzzy logic approach of [Lee72], which allow to treat vague predicates in a truth-functional manner.[3] Our uncertain logic programs have a model-theoretic foundation which is closely related to possibilistic logic treating uncertain information in a non-truth-functional way. While we have presented a faithful embedding of normal logic programs into our uncertain logic programs (with stable semantics), we do not know of any similar one into fuzzy logic programs (the main problem for this seems to be the fuzzy logic evaluation of negation: $C(\neg F) = 1 - C(F)$). Notice also that vagueness and uncertainty are orthogonal concepts, and might be combined in an appropriate framework.

[3] A recent example is [EM95].

Another proposal claiming to subsume uncertainty handling (as well as mutiple-valued and temporal sentences) in logic programming as a special case is the *annotated logic programming* framework of [Sub87, Sub94]. As was already pointed out in [DLP91], however, annotated logic programming is truth-functional and therefore not capable of capturing uncertainty in the information-based semantical perspective of possibilistic logic. Although the notion of a stable model is defined for annotated logic programs in [Sub94], no embeddability result for normal programs is presented.

In [DLP91], the authors present a possibilistic extension of positive logic programs. Since negation is not allowed, the resulting system is monotonic, and normal logic programs cannot be embedded.

8 Conclusion

We have shown how to combine the notion of uncertain infomation with the semantical framework of logic programming, i.e. with minimal and stable Herbrand models. Our notion of uncertainty is derived from the possibility theory of Zadeh, Dubois and Prade which seems to be the natural choice in the context of an information-based semantics (as opposed to a truth-based semantics where probability theory may be more appropriate). We plan to extend our present work towards a combination of uncertainty and extended logic programs where we expect to deal with both certainty- and impossibility-valuated formulas.

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