Modal Logics that Need Very Large Frames

Abstract. The Kuznetsov–Index of a modal logic is the least cardinal μ such that any consistent formula has a Kripke–model of size $\leq \mu$ if it has a Kripke–model at all. The Kuznetsov–Spectrum is the set of all Kuznetsov–Indices of modal logics with countably many operators. It has been shown by Thomason that there are tense logics with Kuznetsov–Index $\beth_{\omega+\omega}$. Futhermore, Chagrov has constructed an extension of $\mathsf{K4}$ with Kuznetsov–Index $\beth_\omega$. We will show here that for each countable ordinal λ there are logics with Kuznetsov–Index $\beth_\lambda$. Furthermore, we show that the Kuznetsov–Spectrum is identical to the spectrum of indices for Π^1_1–theories, which is likewise defined. A particular consequence is the following. If inaccessible (weakly compact, measurable) cardinals exist, then the least inaccessible (weakly compact, measurable) cardinal is also a Kuznetsov–Index.

1. Introduction

Suppose φ is an elementary formula and that φ is consistent with an elementary theory T in a countable language. Then there exists a countable T–model for φ. Furthermore, in any infinite cardinality μ there exists a T–model for φ. For other languages this does not need to hold, for example for second–order logic. Modal logic also has first–order structures, namely Kripke–frames, but the language is a fragment of monadic second order predicate logic. Moreover, modal logics neither necessarily define first–order classes of frames nor is every first–order definable class of frames modally definable (see [1]). The same is true for intermediate logics. Therefore, Hosoi and Ono [9] raised the following question:

> *Do there exist intermediate logics Λ such that Λ is complete but not complete with respect to countable Kripke–frames?*

Shehtman gave a positive answer (see [16]). After showing his solution to A. Kuznetsov, Kuznetsov then asked the following natural question:

> *What is the least cardinal number μ such that any intermediate logic complete with respect to Kripke–frames is also complete with respect to frames of cardinality $\leq \mu$?*

This question remains unsolved. However, the same questions naturally arise also for modal logics. A first example of a logic that is complete but not complete with respect to countable frames was given by Thomason [18] in tense logic. Thomason also established that there are logics Θ_λ for $\lambda < \omega+\omega$ such that Θ_λ is complete, but all its rooted frames have size $\beth_\lambda$. One might suspect that the availability of such logics depends on the number of modal operators. Yet, as Thomason has also shown, any example involving a finite number of operators can be transformed into an example with a single operator. We will improve this in Section 7 showing that any example with countably many operators can be transformed into one using a single modal operator. Since we are dealing only with countable languages, this is the best possible result. We define the *Kuznetsov–Index* of a logic Θ to be the least μ such that any formula which is refutable on a Θ–Kripke–frame is already refutable on a Θ–Kripke–frame of size $\leq \mu$.

The examples constructed using Thomason's method are not transitive. Therefore, to construct logics containing $\mathsf{K4}$ or even Grz of the requested kind is not solved by appealing to polymodal logics. In the intermediate case, an answer was provided by Shehtman [16]. For transitive logics Alexander Chagrov has shown in [4] that there exists a logic Λ containing $\mathsf{K4}$ whose Kuznetsov–Index is $\beth_\omega$.

Both Thomason and Chagrov have indicated that their methods can be extended to higher cardinals. Yet, they did not establish an upper bound on the Kuznetsov–Indices for modal logics. The main result of this paper is that any Π^1_1–definable cardinal number is the Kuznetsov–Index of some logic. It follows that the set of possible Kuznetsov–Indices depends on the set–theoretic assumptions. For example, if inaccessible (weakly compact, measurable) cardinals exist, then the least inaccessible (weakly compact, measurable) cardinal is the Kuznetsov–Index of some monomodal logic. Moreover, we will show that the set of Kuznetsov–Indices is a set of size at most $2^{\aleph_0}$, which is closed under countable limits, the function $\mu \mapsto 2^\mu$ and under the $\beth$–function. It has to be said though that we have not able to determine whether the logics defined in this paper are complete. This is a handicap when discussing the Kuznetsov–Indices of finitely axiomatizable logics. It is easy to see that if Λ has Kuznetsov–Index κ, the completion of Λ also has Kuznetsov–Index κ. But even if Λ is finitely axiomatizable, its completion need not be.

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2. The Kuznetsov–Index

Before we will give examples, it is worthwile discussing the question somewhat. First of all, since the languages we are dealing with are countable, any consistent formula for a logic can be satisfied in a countable algebra. So, the question is not whether for any consistent formula φ there exists a countable model (this is always so) but if there always exists a countable Kripke–model, if a Kripke–model for φ exists at all. The last condition is needed, for there are also incomplete logics. As Chagrov and Zakharyaschev show in [3], there also always exists a general frame with underlying countable Kripke–frame. However, it is easy to see that the question of Hosoi and Ono (for modal logic) is equivalent to the following:

> *Does there exist a complete logic Λ and a Λ–consistent formula φ which has no countable Kripke–model?*

For if Λ is a logic of the first kind and φ has a Kripke–model but has no countable Kripke–model, let Λ^c be the logic of the Kripke–frames of Λ. This logic is complete, and φ is consistent with it. Clearly, Λ^c has the same Kripke–frames as Λ, and so φ has no countable Kripke–model. We call Λ^c the **completion** of Λ.

We define the *Kuznetsov–Index* $Kz(\Lambda)$ of a modal logic Λ as follows.

Definition 2.1. *Let Λ be a modal logic, μ a cardinal number. φ is called μ–**satisfiable** in Λ if it has a Λ–Kripke–model of size $\leq \mu$. Λ is called μ–**complete** if every consistent formula is μ–satisfiable. The **Kuznetsov–Index** of Λ is the least μ such that Λ^c is μ–complete.*

Notice that we have used the completion of Λ in the definition. This has for consequence that the Kuznetsov–Index is always defined even if the logic is incomplete or has no Kripke–frames at all (in which case its Kuznetsov–Index is 0). However, Kuznetsov's original problem concerned the question of finitely axiomatizable logics, and we remark here that Λ^c need not be finitely axiomatizable even if Λ is.

Proposition 2.2.

$$Kz(\Lambda) := sup_{\varphi \notin \Lambda^c}\, inf\{|\mathfrak{F}| : \mathfrak{F} \nvDash \varphi, \mathfrak{F} \models \Lambda, \mathfrak{F} \text{ Kripke–frame}\}$$

For example, if Λ is tabular, its Kuznetsov–Index is finite. The converse also holds, on condition of completeness. If a logic has the finite model property, its Kuznetsov–Index is countable. Here, the converse may be false even if the logic is complete. This suggests to define the modified Kuznetsov–Index:

$$Kz^\star(\Lambda) := inf\{\lambda : \text{ for all } \varphi \notin \Lambda^c \text{ exists } \mathfrak{F} \text{ such that } |\mathfrak{F}| < \lambda, \mathfrak{F} \models \Lambda^c, \mathfrak{F} \nvDash \varphi\}$$

We may therefore modify the previous definition as follows.

Definition 2.3. *Let Λ be a modal logic, μ a cardinal number. φ is called μ–satisfiable* if there is a Λ–Kripke–model for φ which has size $< \mu$. Λ is called μ–**complete*** if every consistent formula is μ–satisfiable*. The **Kuznetsov–Index*** of Λ is the least μ such that Λ^c is μ–complete*.*

For the modified Kuznetsov–Index we have

$$Kz(\Lambda) \leq Kz^{\star}(\Lambda) \leq Kz(\Lambda)^{+}$$

A logic Λ has the finite model property iff it is complete and $Kz^{\star}(\Lambda) \leq \aleph_0$. If Λ is a transitive logic of finite width without the finite model property, then $Kz^{\star}(\Lambda) = \aleph_0$, $Kz(\Lambda) = \aleph_1$, by a result of Kit Fine that all logics of finite width are complete with respect to countable frames (see [8]). Similarly, if Λ is a subframe logic (not necessarily containing $\mathsf{K4}$). (This result is shown in [21], Corollary 3.8.) For the purpose of the next theorem, $sf(\varphi)$ is the set of subformulae of φ.

Proposition 2.4. *$Kz(\Lambda) = Kz^{\star}(\Lambda)$ only if $Kz(\Lambda)$ has cofinality ω. Hence, $Kz^{\star}(\Lambda)$ is either finite, or a successor cardinal or has cofinality ω.*

Proof. Let $\mu := Kz(\Lambda) = Kz^{\star}(\Lambda)$. Consider the functions

$$\begin{array}{rcl} f(\varphi) & := & inf\{|\mathfrak{F}| : \mathfrak{F} \nvDash \varphi, \mathfrak{F} \models \Lambda, \mathfrak{F} \text{ Kripke–frame}\} \\ g(n) & := & sup\{f(\varphi) : \varphi \notin \Lambda^c, |sf(\varphi)| \leq n\} \end{array}$$

Then $\langle g(n) : n \in \omega\rangle$ is an ascending sequence of cardinal numbers $< \mu$. However, the supremum of this sequence is μ, by assumption on μ. Hence, μ has cofinality ω. ☐

In this proof we have defined the function g. This is the (generalization of the) complexity function of [3]. It measures the size of models required to refute formulae of a given length. For logics with the finite model property, this is a function from natural numbers to natural numbers but in general it is a function from natural numbers to cardinal numbers. We just mention that one can also study for compact logics the size of models for infinite sets of formulae. We have not done so here since it is outside the scope of this paper.

Kuznetsov's initial question gives rise to the following two questions:

> *What is the set of cardinal numbers that are the Kuznetsov–Indices of monomodal logics and what is its least upper bound?*

The least upper bound is called the *Löwenheim number* of modal logic. The abovementioned example by Chagrov is a logic with Kuznetsov–Index $\beth_\omega$ and Kuznetsov–Index* $\beth_\omega^+$.

There is an interesting connection between the Kuznetsov–Index for canonical logics and a longstanding conjecture concerning the elementarity of canonical logics.

Conjecture 2.5. *Let Θ be a normal modal logic. If Θ is canonical, then it is complete with respect to some Δ–elementary class of frames.*

The reader is referred to [17] for the background of this conjecture and some attempts to prove it. Suppose now that Θ is canonical. First of all, we note the following.

Proposition 2.6. *Let Θ be canonical. Then $Kz(\Theta) \leq 2^{\aleph_0}$.*

For a proof note that the countably generated free Θ–algebra is countable, and its underlying frame has cardinality $\leq 2^{\aleph_0}$. (So, assuming GCH, the Kuznetsov–Index of a canonical logic can be at most $\aleph_1$.) If Conjecture 2.5 is correct then it will follow from Proposition 4.3 that the Kuznetsov–Index of a canonical logic is $\leq \aleph_0$. It is however clear that if a canonical logic has Kuznetsov–Index $\leq \aleph_0$ it is not necessarily complete with respect to a Δ–elementary class of frames. So, there the following is therefore a weaker conjecture than Conjecture 2.5:

Conjecture 2.7. *Assume that Θ is canonical. Then $Kz(\Theta) \leq \aleph_0$.*

3. Basic Notions and Terminology

Before we begin, let us briefly fix some notation and terminology. We assume some knowledge of set theory, such as cardinal and ordinal numbers and basic arithmetic thereof. Everything needed for our purposes can be found in [7]. As usual, a cardinal is an ordinal number such that no predecessors have the same cardinality. If μ is a cardinal number, μ^+ denotes the successor cardinal and 2^μ the cardinality of the powerset. $\mathrm{cf}(\mu)$, the **cofinality** of μ is the least ordinal λ

such that there exists an ascending sequence $\langle \gamma_{\lambda'} : \lambda' < \lambda \rangle$ whose limit is μ. μ is called **singular** if $\mathrm{cf}(\mu) < \mu$ and **regular** otherwise. The Generalized Continuum Hypothesis (GCH), which is known to be independent of ZFC, is the postulate that $\mu^+ = 2^\mu$. To make the results independent of GCH we make use of the $\beth$–function, which is defined as follows. For an ordinal γ, $\beth_\gamma$ is the cardinal number obtained by iterating exponentiation γ–times, starting at $\aleph_0$.

$$\begin{array}{lcl} \beth_0 & := & \aleph_0 \\ \beth_{\gamma+1} & := & 2^{\beth_\gamma} \\ \beth_\gamma & := & sup\{\beth_\delta : \delta < \gamma\}\ , \quad \gamma \text{ a limit ordinal.} \end{array}$$

Suppose that $\langle T, < \rangle$ is a transitive, irreflexive order with unique least element, such that any branch is well–ordered, every element has no or exactly 2 immediate successors, and all branches have the same well–ordering type. Then we say that $\langle T, < \rangle$ is a **homogeneously binary branching tree**. It is uniquely determined up to isomorphism by the ordering type of one of its branches. The following is well known.

Proposition 3.1. *Let γ be an infinite successor ordinal and $\langle T_\gamma, < \rangle$ be a homogeneously binary branching tree of depth γ. Then $|T_\gamma| = 2^{|\gamma|}$.*

Proof. First, it is clear that if $\gamma \leq \delta$ are ordinals then $|T_\gamma| \leq |T_\delta|$. We may identify the nodes of the binary branching tree $\langle T_\gamma, < \rangle$ with well–ordered sequences of 0's and 1's of length $< \gamma$. Let b_γ denote the set of sequences $\langle x_\delta : \delta < \gamma \rangle$, where $x_\delta \in \{0, 1\}$ for each $\delta < \gamma$. Obviously, $|b_\gamma| = 2^{|\gamma|}$, since each sequence is the (unique) code of a subset of γ. Now, two cases arise. (1) γ is a limit ordinal. Then $|T_\gamma| = |\bigcup_{\delta<\gamma} b_\delta| = \sum_{\delta<\gamma} |b_\delta|$. (2) $\gamma = \gamma' + 1$, γ infinite. Then $|T_\gamma| = |\bigcup_{\delta \leq \gamma'} b_\delta| \geq |b_{\gamma'}| = 2^{|\gamma'|}$. The other inequality is established as follows. By (1) and (2) we get $|b_\gamma| \leq 2^\gamma$ for all infinite γ. Hence $|T_\gamma| \leq |\gamma| \cdot 2^{|\gamma'|} \leq 2^{|\gamma'|}$, by elementary cardinal arithmetic. So, $|T_\gamma| = 2^{|\gamma'|}$. Since $|\gamma| = |\gamma'|$ the claim follows. $\square$

The cardinalities for γ a limit ordinal are much harder to establish, but not needed in sequel. For example, if the branches have well–order type ω, the tree is countable, but if the well–order type γ is at least $\omega + 1$ and countable, then $|T_\gamma| = 2^{\aleph_0}$.

The present paper assumes a fair amount of knowledge in modal logic. For background in modal logic we refer to [13], in which all notions relevant to this paper are explained. We assume that the reader knows the systems $\mathsf{S5}$ and G and has some understanding of tense logic. We will consider not only modal logics of a single operator, but in fact logics with arbitrarily many operators; we only require that the set O of basic operators is countable. This ensures that the language (the set of well–formed formulae) is a countable set. A **modal logic** over O is a normal polymodal logic using the set O of modal operators. If $|O| = \kappa$, we also say that Λ is a κ–**modal logic**. If $\kappa = 1$ we call Λ a **monomodal logic**. A **Kripke–frame** for Λ is a pair $\langle F, R \rangle$ where F is a set (possibly empty) and $R : O \to F \times F$ a function assigning to each $\Box \in O$ its accessibility relation, $R(\Box)$. Alternatively, when $O = \kappa$, a cardinal number, a frame is a pair $\langle F, \langle \lhd_j : j \in \kappa \rangle \rangle$, where $\lhd_j \subseteq F \times F$ for each $j \in \kappa$. (Often, we will use ordinal numbers rather than cardinals to index the modal operators. This makes life easier. We also write $j < \kappa$ in place of $j \in \kappa$.) A **(generalized) frame** is a triple $\langle F, R, \mathbb{F} \rangle$ such that $\langle F, R \rangle$ is a Kripke–frame and $\mathbb{F} \subseteq \wp(F)$ a set closed under relative complement, intersection, union and

$$A \mapsto \{x : \text{ for all } y \text{ such that } x\ R(\Box)\ y : y \in A\}$$

where $\Box$ is a modal operator of the language. The notions of valuation and satisfaction in a (Kripke-)frame are defined as usual. The operator $\Diamond$ defined by $\Diamond\varphi := \neg\Box\neg\varphi$ is the usual dual operator. We call an operator $\Box'$ a **tense dual** of $\Box$ (with respect to a logic Λ) if $p \to \Box\Diamond' p, p \to \Box'\Diamond p \in \Lambda$. If $\Box'$ is a tense dual of $\Box$ with respect to Λ, then in any Λ–Kripke–frame $\mathfrak{F}$ we have $R(\Box) = R(\Box')^{\smile}$, where for a relation R we denote by $R^{\smile}$ the converse of R. Given a logic Λ and a set X of modal formulae, $\Lambda \oplus X$ denotes the least normal modal logic containing Λ and X. Furthermore, given two modal logics Λ and Θ with disjoint sets of operators, $\Lambda \otimes \Theta$ is the least logic in the union of the languages, which contains both Λ and Θ. (If Λ and Θ share some modal operators, they are suitably renamed to make the sets of operators disjoint.) We note that as a consequence of the theorem of [14] we obtain

Lemma 3.2. *Let μ and ν be infinite. Suppose that Λ and Θ are μ–complete*. Then $\Lambda \otimes \Theta$ is μ–complete* as well. So, if $Kz^\star(\Lambda) = \mu$ and $Kz^\star(\Theta) = \nu$, then $Kz^\star(\Lambda \otimes \Theta) = max\{\mu, \nu\}$.*

Proof. The construction of [14] is as follows. Given a frame $\mathfrak{F}_0$ for Λ, we let grow a Θ–frame at each world of $\mathfrak{F}_0$, and obtain a frame $\mathfrak{F}_1$. Next we let grow a Λ–frame at each node of $\mathfrak{F}_1$. And so on. We need to iterate this finitely often. Each of the frames can be chosen $< \xi$, where $\xi := max\{\mu, \nu\}$. Hence, at each stage the frame has size $< \xi$. Since we iterate finitely often, the entire frame has size $< \xi$. □

We remark that if μ and ν are finite then $max\{\mu, \nu\} \leq Kz^\star(\Lambda \otimes \Theta) \leq \aleph_0$. In both cases, the inequality may be strict. To ease the manufacturing of logics with special Kuznetsov–Index we note the following useful fact.

Lemma 3.3. *There exists a logic with Kuznetsov–Index* μ^+ iff there exists a complete logic Θ and a formula which is μ–satisfiable in Θ but not μ–satisfiable*.*

Proof. Let Λ have Kuznetsov–Index* μ^+. Then there is a φ such that there is no model based on a frame of cardinality $< \mu$, but there is a model based on some $\mathfrak{F}$ of cardinality μ. Put $\Theta := \mathsf{Th}\,\mathfrak{F}$. This logic is obviously complete; and it has Kuznetsov–Index* $\leq \mu^+$, since any consistent formula can be satisfied on $\mathfrak{F}$. By the fact that $\Theta \supseteq \Lambda$ and $\varphi \notin \Theta$, no Θ–Kripke–model for φ has less than μ worlds. Hence $Kz^\star(\Theta) = \mu^+$. Conversely, assume that Λ is such that a formula φ exists which is μ–satisfiable but not μ–satisfiable*. Take a Kripke–frame $\mathfrak{F}$ such that $\mathfrak{F} \nvDash \neg\varphi$. Put $\Theta := \mathsf{Th}\,\mathfrak{F}$. Then Θ has Kuznetsov–Index* μ^+. □

Lemma 3.4. *Let μ be a limit cardinal. There exists a logic with Kuznetsov–Index* μ iff there exists a complete logic Θ and an ascending sequence $\langle \lambda_i : i \in \omega \rangle$ of cardinals with limit μ and a sequence $\langle \varphi_i : i \in \omega \rangle$ of formulae such that for each $i \in \omega$ φ_i is λ_i–satisfiable in Θ but not λ_i–satisfiable*.*

The proof is immediate.

In [6], Maarten de Rijke has introduced the so–called **difference operator**. He uses D to denote this operator, but we follow our general practice and write $[\neq]$ for the box–like analogon and $\langle \neq \rangle$ for its dual. The intended semantics for this operator is that of the difference, that is, we want to have $R([\neq]) = \{\langle x, y \rangle : x \neq y\}$. For well–known reasons this is impossible, so it is required to hold only for rooted frames. It is not possible to define the logic of the difference operator in such a way that the intended Kripke–frames are the only Kripke–frames of the logic. There is a way, however, to achieve this (see [15]). Namely, instead of the difference operator take a pair of modal operators, which are tense duals of each other and look in both directions of the well–order. In general, the construction is as follows. Let Λ be a κ–modal logic. Let WO be the tense logic in two operators, $\boxplus$ $(:= \Box_0)$ and $\boxminus$ $(:= \Box_1)$, which satisfy the following postulates. (The axiomatization is not independent. Some of the axioms can be dropped from the list.)

$$
\begin{array}{rl}
\mathsf{WO} := & \mathsf{K}_2 \\
\oplus & p \to \boxplus\neg\boxminus\neg p \\
\oplus & p \to \boxminus\neg\boxplus\neg p \\
\oplus & \boxplus p \to \boxplus\boxplus p \\
\oplus & \boxminus p \to \boxminus\boxminus p \\
\oplus & \boxminus(\boxminus p \to p) \to \boxminus p \\
\oplus & \neg\boxplus\boxminus p \to \neg p \vee \neg\boxplus p \vee \neg\boxminus p \\
\oplus & \neg\boxminus\boxplus p \to \neg p \vee \neg\boxminus p \vee \neg\boxplus p
\end{array}
$$

Lemma 3.5. WO *is the tense logic of well–orders, where $R(\boxplus) = <$ and $R(\boxminus) = >$.*

The proof is straightforward. WO is clearly a tense logic, and so $R(\boxminus) = R(\boxplus)^{\smile}$. $R(\boxminus)$ is transitive and satisfies G, whence the Kripke–structures may not contain any infinite downgoing chains. Both $R(\boxminus)$ and $R(\boxplus)$ are linear. By a result of Frank Wolter [19] this logic is complete with respect to the well–orders. So, WO is the desired logic of well–orders.

Definition 3.6. *Let Λ be a κ–modal logic. The $\kappa+2$–modal logic Λ^{wo} is defined by*

$$\Lambda^{wo} := \Lambda \otimes \mathsf{WO} \oplus \{p \wedge \boxminus p \wedge \boxplus p. \to .\Box_j p : j < \kappa\}$$

Lemma 3.7. *The Kripke–frames of Λ^{wo} are the frames $\langle F, \langle \lhd_j : j < \kappa+2\rangle\rangle$ such that $\langle F, \langle \lhd_j : j < \kappa\rangle\rangle$ is a Λ–frame, and $\lhd_\kappa$ is a well–order on F, whose symmetric and reflexive closure contains all $\lhd_j$, $j < \kappa$, and $\lhd_{k+1} = \lhd_\kappa^{\smile}$. In particular, Λ^{wo} is conservative over Λ if Λ is complete.*

By a general result on complete subframe logics (see [21]), if a subframe logic is complete it is actually complete with respect to countable frames. Hence, $Kz^\star(\mathsf{WO}) = \aleph_1$, since the logic of well–orders fails to have the finite model property. (To see that, notice that the formula $\boxplus(\boxplus p \to p) \to \boxplus p$ is not valid in WO, since well–orders may possess infinite ascending chains. However, no finite frame refutes this formula.)

Lemma 3.8. *Let Λ and Θ be α–modal and β–modal languages, respectively, and let $\alpha \leq \beta$. Suppose that Θ is conservative over Λ. Then $Kz^\star(\Theta) \geq Kz^\star(\Lambda)$ and $Kz(\Theta) \geq Kz(\Lambda)$.*

Lemma 3.9. *Suppose that $\mu = Kz^\star(\Lambda) > \aleph_0$. Then $Kz^\star(\Lambda^{wo}) \geq \mu$. Moreover, let Θ be the logic of all Kripke–frames of Λ^{wo} of cardinality $< \mu$. Then Θ is complete and $Kz^\star(\Theta) = \mu$.*

Proof. Let $\lambda := Kz^\star(\Lambda^{wo})$. We show that $\lambda \geq \mu$. The reader may reflect on the fact that we can assume without loss of generality that Λ is complete. Then Λ^{wo} is conservative over Λ and so by Lemma 3.8 $\lambda \geq \mu$. For the second claim, let $\kappa := Kz^\star(\Theta)$. By definition of Θ, $\kappa \leq \mu$. But Θ is also conservative over Λ and so $\kappa \geq \mu$. □

This lemma will be quite useful later on. The difference operator is now easily definable:

$$[\neq]\varphi := \boxplus\varphi \wedge \boxminus\varphi$$

It is to be borne in mind that $R([\neq]) = \{\langle x, y\rangle : x \neq y\}$ only if $\mathfrak{F}$ is rooted.

Let us define the following sets

$$\begin{array}{lcl}
\mathbb{K}_\alpha & := & \{Kz(\Theta) : \Theta \text{ an } \alpha\text{–modal logic}\} \\
\mathbb{K}^\star_\alpha & := & \{Kz^\star(\Theta) : \Theta \text{ an } \alpha\text{–modal logic}\} \\
\mathbb{K}^f_\alpha & := & \{Kz(\Theta) : \Theta \text{ a finitely axiomatizable } \alpha\text{–modal logic}\} \\
\mathbb{K}^{\star f}_\alpha & := & \{Kz^\star(\Theta) : \Theta \text{ a finitely axiomatizable } \alpha\text{–modal logic}\}
\end{array}$$

We call these sets the α–**Kuznetsov–Spectrum** and the α–**Kuznetsov–Spectrum***, and the **finitary** α–**Kuznetsov–Spectrum** and **finitary** α–**Kuznetsov–Spectrum***, respectively. Finally, define

$$\begin{array}{lclclcl}
\rho_\alpha & := & sup\,\mathbb{K}_\alpha & \qquad & \rho^f_\alpha & := & sup\,\mathbb{K}^f_\alpha \\
\rho^\star_\alpha & := & sup\,\mathbb{K}^\star_\alpha & & \rho^{\star f}_\alpha & := & sup\,\mathbb{K}^{\star f}_\alpha
\end{array}$$

We shall call ρ_α the **Löwenheim–number** and ρ^f_α the **finitary Löwenheim number** of α–modal logic. It will be established that $\rho_\alpha = \rho^\star_\alpha$ and $\rho^f_\alpha = \rho^{\star f}_\alpha$, so that no name needs to be given to the other numbers. ρ_α (ρ^f_α) is the least cardinality such that for any (finitely axiomatizable) α–modal logic Θ and any consistent formula φ, if φ has a Kripke–model in Θ, then it has a Kripke–model of size $\leq \rho_\alpha$ ($\leq \rho^f_\alpha$). Similarly for $\rho^\star_\alpha$ and $\rho^{\star f}_\alpha$. The following is easy to establish.

Proposition 3.10. *Assume $0 < \alpha, \beta < \aleph_1$.*

1. $\mathbb{K}^f_\alpha \subseteq \mathbb{K}_\alpha$.
2. $\mathbb{K}^f_\alpha$ *is a set of cardinality* $= \aleph_0$.
3. $\mathbb{K}_\alpha$ *is a set of cardinality* $\leq 2^{\aleph_0}$.
4. $\mathbb{K}_\alpha$ *contains all finite cardinal numbers and* $\aleph_0$.
5. $\mathbb{K}^f_{\aleph_0} = \{0\}$.

6. $\mathbb{K}^f_\alpha$ *contains all finite cardinal numbers and* $\aleph_0$ *for finite* α.
7. *If* $\alpha < \beta$ *then* $\mathbb{K}_\alpha \subseteq \mathbb{K}_\beta$ *and* $\rho_\alpha \leq \rho_\beta$.

Similarly for $\mathbb{K}^{\star(f)}_\alpha$ *and* $\rho^{\star(f)}_\alpha$.

Notice that if α is infinite, then a finitely axiomatizable extension of K_α is necessarily inconsistent. Thus $\mathbb{K}^f_{\aleph_0} = \{0\}$. The last claim is shown as follows. Let Λ be an α–modal logic with Kuznetsov–Index μ. Then let Θ be a modal logic based on one point and with operators $\Box_i$, $\alpha \leq i < \beta$. Then $\Lambda \otimes \Theta$ has the same Kuznetsov–Index as Λ.

4. A First Example

Our first example is the logic of the line of real numbers in the language of tense logic and the difference operator. To motivate the example and to show the validity of our claims, we will build up this example starting with the modal logic of the real line. Therefore, consider first the real line $\langle \mathbb{R}, < \rangle$ as a Kripke–frame for a monomodal logic. This logic is $\mathsf{D4.3} \oplus \boxplus^2 p \to \boxplus p$. This is the same as the modal theory of $\langle \mathbb{Q}, < \rangle$. Hence, its Kuznetsov–Index is $\leq \aleph_0$. Now adjoin a tense dual, $\boxminus$. Then $R(\boxminus) = R(\boxplus)^\smile$, and therefore we can regard $\langle \mathbb{R}, < \rangle$ and $\langle \mathbb{Q}, < \rangle$ in a natural way as Kripke–frames for this language. Now we can distinguish the theory of the reals from the theory of the rational numbers. Call a **gap** in a linearly ordered set $\langle A, < \rangle$ a pair of open intervals B and C such that $B \cap C = \varnothing$ and $B \cup C = A$. It has been observed by Frank Wolter in [20] that the property of not possessing a gap can be expressed axiomatically in tense logic. It amounts to the property of not containing the linear reflexive frame with two points. So, the tense logic of the real line is a splitting of the theory of dense linear orders without end points by a two point frame. However, as has been shown by Robert Bull in [2], the tense logic of the real line has the finite model property. The problem is that this logic admits frames in which $R(\boxplus)$ is not irreflexive. If it were, no countable orders can exist. For then in a Kripke–frame $R(\boxplus)$ would be an irreflexive, dense linear order without end points, which is complete. Now we add two more operators. These two operators serve to define the difference operator. The structures over which we now talk are triples $\langle A, <, \sqsubset \rangle$, where $\langle A, < \rangle$ is a dense linear order without end points and gaps, and $\sqsubset$ is a well–order on A. It is now easy to see that this logic has no countable frames. To that effect notice the following. The formula $[\neq]p \to \boxplus p$ is an axiom of the logic. Therefore, the relation corresponding to $\boxplus$ is irreflexive. We conclude that with this axiom, the logic has no countable frames. Hence, the Kuznetsov–Index of this logic is exactly $2^{\aleph_0}$ since any consistent formula is satisfiable in $\mathbb{R}$.

Theorem 4.1. *Let* Θ *be the logic of structures* $\langle \mathbb{R}, <, \sqsubset \rangle$ *in the language of tense logic for both orders, where* $\langle \mathbb{R}, < \rangle$ *is the real line and* $\langle \mathbb{R}, \sqsubset \rangle$ *a well–order. Then* Θ *has no countable models. In particular,* $Kz(\Theta) = 2^{\aleph_0}$.

The resulting logic is a 4–modal logic. To get a monomodal logic with these properties we invoke the simulation theorem from [13]. This theorem states that for every finite number k there is an isomorphism $\Theta \mapsto \Theta^s$ from the lattice of k–modal logics onto an interval in the lattice of monomodal logics such that the property of completeness is left invariant. It is easy to see that $Kz(\Theta^s) = k \cdot Kz(\Theta) + k - 1$.

Theorem 4.2. *There exists a normal monomodal logic with Kuznetsov–Index* $2^{\aleph_0}$.

Now what happens if we require that Θ is canonical? We have no answer to the question. But there is one on condition that Θ is Δ–elementary. To define that notion properly, let L_α be the first–order language based on binary relation symbols R_i, $i < \alpha$, no constants and no function symbols. A class $\mathcal{K}$ of Kripke–frames is called **elementary** if there is a sentence $\gamma \in L_\alpha$ such that $\mathfrak{F} \in \mathcal{K}$ iff $\mathfrak{F} \models \gamma$. An intersection of elementary classes is called Δ–elementary. An α–modal logic Θ is **elementary** (**Δ–elementary**) if its class of Kripke-frames is elementary (Δ–elementary).

Proposition 4.3. *Let* Θ *be a* Δ*–elementary logic based on a countable language. Then it has Kuznetsov–Index* $\leq \aleph_0$.

There are two proofs, one using elementary expansions and the other using modal expansions. We will present both. If Θ is elementary, its class of frames is characterized by some countable set $T \subseteq L_\alpha$. Now adjoin to L_α a unary relational constant C_i for each $i < \omega$. Call the expansion L_α^+. Following [1], define a translation of φ by

$$\begin{array}{lcl} p_i^\dagger & := & \mathsf{C}_i(x) \\ (\neg\varphi)^\dagger & := & \neg\varphi^\dagger \\ (\varphi \wedge \psi)^\dagger & := & \varphi^\dagger \wedge \psi^\dagger \\ (\Box\varphi)^\dagger & := & (\forall y)(x\ R(\Box)\ y \rightarrow \varphi^\dagger[y/x]) \end{array}$$

In the last clause y is a variable not already occurring in $\varphi^\dagger$. The following is clear.

Lemma 4.4. *For every α–modal Kripke–frame $\mathfrak{F}$: $\mathfrak{F} \nvDash \varphi$ iff for some L_α^+–expansion $\mathfrak{F}^+$: $\mathfrak{F}^+ \nvDash \varphi^\dagger$.*

Now: $\Theta \nvDash \varphi$ iff there exists a Kripke–frame $\mathfrak{F}$ for Θ such that $\mathfrak{F} \nvDash \varphi$ iff there exists an L_α–structure $\mathfrak{F}$ such that $\mathfrak{F} \models T$ and for some L_α^+–expansion $\mathfrak{F}^+$: $\mathfrak{F}^+ \models T$ and $\mathfrak{F}^+ \nvDash \varphi^\dagger$ iff there exists a countable L_α^+–structure $\mathfrak{G}^+$ such that $\mathfrak{G}^+ \models T$ and $\mathfrak{G}^+ \nvDash \varphi^\dagger$ iff for some countable Θ–Kripke–frame $\mathfrak{G}$: $\mathfrak{G} \nvDash \varphi$.

The second proof is intrinsic (and actually more general). We eliminate the variables in φ by introducing a new modal operator $\boxtimes$. We substitute in φ the variable p_i uniformly by

$$\chi_i := \neg \boxtimes \neg(\boxtimes^{i+1} \bot \wedge \neg \boxtimes^i \bot) \ ,$$

for all $i < \omega$. Denote the result of this substitution by $\varphi^\ddagger$.

Lemma 4.5. *$\varphi \in \Theta$ iff $\varphi^\ddagger \in \Theta \otimes \mathsf{K}$.*

Proof. If $\varphi \in \Theta$ then $\varphi \in \Theta \otimes \mathsf{K}$ and so $\varphi^\ddagger \in \Theta \otimes \mathsf{K}$. So, the other direction needs proof. Suppose that $\varphi \notin \Theta$. Then there exists a model $\langle \mathfrak{F}, \beta, u \rangle \models \neg\varphi$, based on a generalized frame $\langle F, R, \mathbb{F} \rangle$. We construct a $\Theta \otimes \mathsf{K}$–frame $\mathfrak{F}^+$, a valuation β^+ and a point u^+ such that $\langle \mathfrak{F}^+, \beta^+, u^+ \rangle \models \neg\varphi^\ddagger$. Put $F^+ := F \times (\{\star\} \cup \omega)$ and for each basic modality $\Box_i$ of Θ:

$$R^+(\Box_i) := \{\langle\langle x, j\rangle, \langle y, j\rangle : x\ R(\Box_i)\ y, j \in \{\star\} \cup \omega\}$$

Next, for the additional modality put

$$R^+(\boxtimes) := \left\{ \begin{array}{ll} & \{\langle\langle x, \star\rangle, \langle x, j\rangle\rangle : \langle \mathfrak{F}, \beta, x\rangle \models p_j\} \\ \cup & \{\langle\langle x, j+1\rangle, \langle x, j\rangle\rangle : j \in \omega\} \end{array} \right.$$

And finally, let $\mathbb{F}^+$ consist of all unions of sets of the form $a \times \{i\}$, $i \in \{\star\} \cup \omega$, where $a \in \mathbb{F}$. It is straightforward to check that this is a generalized frame. Furthermore, if $\mathfrak{F}^+$ is restricted to the modalities of Θ, it is a union of copies of $\mathfrak{F}$, and so $\mathfrak{F}^+ \models \Theta$. This shows that $\mathfrak{F}^+ \models \Theta \otimes \mathsf{K}$. Next, $\langle \mathfrak{F}^+, \langle x, \star\rangle\rangle \models \chi_j$ iff $\langle \mathfrak{F}, \beta, x\rangle \models p_j$. It follows by an easy induction that $\langle \mathfrak{F}^+, \langle x, \star\rangle\rangle \models \varphi^\ddagger$ iff $\langle \mathfrak{F}, \beta, x\rangle \models \varphi$. This establishes the claim. ☐

Now, if χ is constant, $\chi^\dagger$ is actually an L_α–sentence. So, if the class of Θ is characterized by T, the class of Θ–frames refuting $\varphi^\ddagger$ is characterized by $T \cup \{\neg(\varphi^\ddagger)^\dagger\}$. Hence the proof is completed by the following observation, which is easy to prove. (Or see [14] for a proof.)

Lemma 4.6. *Suppose that Θ is a canonical modal logic. Then $\Theta \otimes \mathsf{K}$ is also canonical. Moreover, if Θ is elementary, so is $\Theta \otimes \mathsf{K}$.*

We will draw from the proof two simple consequences.

Lemma 4.7. *Let μ be infinite. Suppose that there exists a logic Θ with $Kz^\star(\Theta) = \mu^+$. Then there exists a logic $\Theta^\bullet$ with $Kz^\star(\Theta^\bullet) = \mu^+$ and a constant formula χ which is μ–satisfiable but not μ–satisfiable*.*

Proof. By Lemma 3.3 there is a formula φ which is μ–satisfiable but not μ–satisfiable* in Θ. Now let $\Theta^\bullet := \Theta \otimes \mathsf{K}$ and $\chi := \varphi^\ddagger$, defined above. By Lemma 3.2. this logic is complete and $Kz^\star(\Theta_1) = \mu^+$. χ has a model of size μ in $\Theta^\bullet$ but no model of size $< \mu$. ☐

Lemma 4.8. *Let μ be infinite. Suppose that there exists a logic Θ with $Kz^\star(\Theta) = \mu^+$. Then there exists a complete logic $\Theta^\bullet$ with $Kz^\star(\Theta^\bullet) = \mu^+$ which has no frames of cardinality $< \mu$.*

Proof. By the Lemma 4.7 there exists a logic Θ with $Kz^\star(\Theta) = \mu^+$ and a constant χ which is not satisfiable in frames of cardinality $< \mu$. By Lemma 3.9, we may without loss of generality also assume that the difference operator is in the language of Θ. Put $\Theta^\heartsuit := \Theta \oplus \chi \vee \langle\neq\rangle\chi$. In this logic, $\top$ is μ–satisfiable but not μ–satisfiable*. Let $\Theta^\bullet$ be the logic of the $\Theta^\heartsuit$–frames of cardinality μ. Then $\Theta^\bullet$ has Kuznetsov–Index* μ^+. Moreover, $\top$ is μ–satisfiable but not μ–satisfiable*. This means that there exists no $\Theta^\bullet$–frame of cardinality $< \mu$. □

5. Binary Branching Trees

In this and the next section we shall construct modal logics with countably many operators whose Kuznetsov–Index is exactly $\beth_\lambda$, where λ is a countable ordinal. Let us take five modal operators, $\boxminus_0$, $\boxminus_1$, $\boxminus$, $\blacksquare\!\!-$ and $-\!\!\blacksquare$, such that the following holds.

1. If $x\ R(\boxminus_0)\ y_0, y_1$ then $y_0 = y_1$.
2. If $x\ R(\boxminus_1)\ y_0, y_1$ then $y_0 = y_1$.
3. $R(\boxminus) = (R(\boxminus_0) \cup R(\boxminus_1))^\smile$.
4. $R(\blacksquare\!\!-)$ contains the transitive closure of $R(\boxminus_0) \cup R(\boxminus_1)$.
5. $R(-\!\!\blacksquare) = R(\blacksquare\!\!-)^\smile$.
6. $R(-\!\!\blacksquare)$ is locally linear and has no infinite ascending chains.
7. If $x\ R(\blacksquare\!\!-)\ y$, $x\ R(\boxminus_0)\ z_0$ and $x\ R(\boxminus_1)\ z_1$ then either $z_0\ R(\blacksquare\!\!-)\ y$ does not obtain or $z_1\ R(\blacksquare\!\!-)\ y$ does not obtain.
8. If $x\ R(\blacksquare\!\!-)\ y_0, y_1$ then either
   (a) $y_0\ R(\blacksquare\!\!-)\ y_1$ or
   (b) $y_1\ R(\blacksquare\!\!-)\ y_0$ or
   (c) $y_0 = y_1$ or
   (d) there exists an x' such that $x = x'$ or $x\ R(\blacksquare\!\!-)\ x'$ and for no $R(\blacksquare\!\!-)$–successor w of x', both $w\ R(\blacksquare\!\!-)\ y_0$ and $w\ R(\blacksquare\!\!-)\ y_1$ obtain.

(A relation R is **locally linear** if $x\ R\ y_0, y_1$ implies $y_0 = y_1$, $y_0\ R\ y_1$ or $y_1\ R\ y_0$.) With the exception of the last two conditions it is not difficult to see that these conditions can be captured by modal axioms. However, (7) and (8) are quite problematic. For them we must actually introduce the difference operator, $[\neq]$ (which we will thereafter eliminate by two tense duals using a well–order, as above). Note the following fact, which is easy to prove.

Lemma 5.1. *Put* $\mathrm{n}(p) := p \wedge [\neq]\neg p$. *Let $\mathfrak{F}$ be a rooted Kripke–frame. Then* $\langle \mathfrak{F}, \beta, x\rangle \models \mathrm{n}(p)$ *iff* $\beta(p) = \{x\}$.

Lemma 5.2. *Let $\mathfrak{F}$ be a Kripke–frame satisfying (1) – (6). Then $\mathfrak{F} \models (7)$ iff*

$$\mathfrak{F} \models \blacklozenge\!\!-\mathrm{n}(p) \to (\boxminus_0 \blacksquare\!\!-\neg p \vee \boxminus_1 \blacksquare\!\!-\neg p)$$

Proof. Assume that $\mathfrak{F}$ satisfies the modal formula. Suppose that $x\ R(\blacksquare\!\!-)\ y$, $x\ R(\boxminus_0)\ z_0$ and $x\ R(\boxminus_1)\ z_1$. Put $\beta(p) := \{y\}$. Then $\langle \mathfrak{F}, \beta, x\rangle \models \blacklozenge\!\!-\mathrm{n}(p)$. Now $x \models \boxminus_0 \blacksquare\!\!-\neg p \vee \boxminus_1 \blacksquare\!\!-\neg p$, from which either $x \models \boxminus_0 \blacksquare\!\!-\neg p$ or $x \models \boxminus_1 \blacksquare\!\!-\neg p$. Assume the first. Then $z_0 \models \blacksquare\!\!-\neg p$ and so $z_0\ R(\boxminus)\ y$ does not hold. Assume the second. Then $z_1 \models \blacksquare\!\!-\neg p$ and so $z_1\ R(\blacksquare\!\!-)\ y$ does not hold. So, $\mathfrak{F}$ satisfies (7). Now assume conversely that $\mathfrak{F}$ satisfies (7). Assume that $\langle \mathfrak{F}, \beta, x\rangle \models \blacklozenge\!\!-\mathrm{n}(p)$. Then $\beta(p) = \{y\}$ for some y such that $x\ R(\blacksquare\!\!-)\ y$. Pick z_0 and z_1 such that $x\ R(\boxminus_0)\ z_0$ and $x\ R(\boxminus_1)\ z_1$. Then either $z_0\ R(\blacksquare\!\!-)\ y$ does not hold, and so $z_0 \models \blacksquare\!\!-\neg p$, or $z_1\ R(\blacksquare\!\!-)\ y$ does not hold, and so $z_1 \models \blacksquare\!\!-\neg p$. It follows that $x \models \Diamond_0 \blacksquare\!\!-\neg p \vee \Diamond_1 \blacksquare\!\!-\neg p$. But from (1) and (2) we deduce that also $x \models \boxminus_1 \blacksquare\!\!-\neg p \vee \boxminus_1 \blacksquare\!\!-\neg p$. So, $\mathfrak{F}$ satisfies the modal formula above. □

Likewise there is a modal counterpart of the last postulate. For the purpose of its definition let $\blacklozenge\!\!-^{\leq 1}\varphi := \varphi \vee \blacklozenge\!\!-\varphi$.

Lemma 5.3. *Let $\mathfrak{F}$ be a Kripke–frame satisfying (1) – (6). Then $\mathfrak{F} \models (8)$ iff*

$$\mathfrak{F} \models \blacklozenge\!\!-\mathrm{n}(p) \wedge \blacklozenge\!\!-\mathrm{n}(q). \rightarrow .\blacklozenge\!\!-(p \wedge \blacklozenge\!\!-q) \vee \blacklozenge\!\!-(q \wedge \blacklozenge\!\!-p) \vee \blacklozenge\!\!-(p \wedge q) \vee \blacklozenge\!\!-^{\leq 1}\blacksquare\!\!-(\blacksquare\!\!-\neg p \vee \blacksquare\!\!-\neg q)$$

Proof. Call the modal formula ζ. Assume that $\mathfrak{F} \models (8)$. Let β be such that $\langle \mathfrak{F}, \beta, x\rangle \models \blacklozenge\!\!-\mathrm{n}(p); \blacklozenge\!\!-\mathrm{n}(q)$. Then we have $\beta(p) = \{y_0\}$ and $\beta(q) = \{y_1\}$ for some y_0 and y_1 with $x\ R(\blacksquare\!\!-)\ y_0, y_1$. Now, $\mathfrak{F} \models \zeta$ and so either (a) $x \models \blacklozenge\!\!-(p \wedge \blacklozenge\!\!-q)$, in which case $y_0\ R(\blacksquare\!\!-)\ y_1$ or (b) $x \models \blacklozenge\!\!-(q \wedge \blacklozenge\!\!-p)$, in which case $y_1\ R(\blacksquare\!\!-)\ y_0$ or (c) $x \models \blacklozenge\!\!-(p \wedge q)$, in which case $y_0 = y_1$, or (d) $x \models \blacklozenge\!\!-^{\leq 1}\blacksquare\!\!-(\blacksquare\!\!-\neg p \vee \blacksquare\!\!-\neg q)$. If (d) obtains, there is an x' such that $x = x'$ or $x\ R(\blacksquare\!\!-)\ x'$ and $x' \models \blacksquare\!\!-(\blacksquare\!\!-\neg p \vee \blacksquare\!\!-\neg q)$. This means that for any $R(\blacksquare\!\!-)$–successor w of x', either $w\ R(\blacksquare\!\!-)\ y_0$ does not obtain or $w\ R(\blacksquare\!\!-)\ y_1$ does not obtain. This is as claimed. The converse is as straightforward. □

Call Π the logic of all frames satisfiying (1) – (8). It is now important to note that the rooted Kripke–frames for Π are binary branching trees. Moreover, suppose that $p = \langle x_i : i < \omega\rangle$ is a path, that is, $x_i\ R(\boxminus_0)\ x_{i+1}$ or $x_i\ R(\boxminus_1)\ x_{i+1}$ for all $i < \omega$ and suppose that there exists a supremum y_p of this path in $R(\blacksquare\!\!-)$. This supremum does not need to exist, but it exists as soon as the path has an upper cover with respect to $R(\blacksquare\!\!-)$. It is unique, by the last postulate. For any two incomparable $R(\blacksquare\!\!-)$–successors must at some point of the path lead up to distinct successors. Now take another path q starting at x_0. Suppose that it too has a supremum, y_q. Then there is an $i < \omega$ such that the point x_{i+1} is not in the path q. We then have that y_q is not a $R(\blacksquare\!\!-)$–successor of x_{i+1} or in fact of any x_j, $j > i$. Hence, any two paths starting at the same point define a different set of suprema. The same fact can be shown for ascending chains for $R(\blacksquare\!\!-)$. Therefore, the Π–frames really are binary branching trees.

Lemma 5.4. *Let $\mathfrak{F}$ be a Kripke–frame for Π. Then $\mathfrak{F}$ is a binary branching tree whose paths are well–ordered.*

Finally, we will arrange it that the models for the logic are not only binary branching trees but binary branching trees in which every path has the same well–ordering type. To do this we introduce a new modal operator, $[\circ]$. It shall satisfy $\mathsf{S5}$ and the intention is that $x\ R([\circ])\ y$ whenever x and y are of the same level in the tree. We write $x \circ y$ iff $x\ R([\circ])\ y$. This can be achieved by the following postulates.

Lemma 5.5. *Suppose $\mathfrak{F}$ is a rooted Π–Kripke–frame. Suppose further that $R([\circ])$ is a relation on F such that $\mathfrak{F} \models \nu$, where*

$$\nu := \mathrm{n}(p) \wedge \langle\neq\rangle\,\mathrm{n}(q) \rightarrow (\langle\circ\rangle q \leftrightarrow -\!\!\blacksquare\,\langle\circ\rangle\,\blacklozenge\!\!-(q \wedge -\!\!\blacksquare\,\langle\circ\rangle\,\blacklozenge\!\!-p))$$

Then $x \circ y$ iff x and y have the same depth in the binary branching tree $\langle F, R(\blacksquare\!\!-)\rangle$.

Proof. By order induction. Assume that for every $\delta < \gamma$ the claim holds. We aim to show that it holds for γ. The case $\gamma = 0$ is settled by assumption that $\mathfrak{F}$ is rooted. If $\gamma > 0$, let x be of depth γ. Then x has $R(-\!\!\blacksquare)$–successors. Assume that $x \circ y$ but y has depth $\gamma' \neq \gamma$. Without loss of generality we may assume that $\gamma' > \gamma$. Put $\beta(p) := \{x\}$ and $\beta(q) := \{y\}$. Then the antecedent of ν is true, since p and q hold at exactly one point. Assume $x \circ y$. Then $x \models \langle\circ\rangle q$, and so $x \models -\!\!\blacksquare\,\langle\circ\rangle\,\blacklozenge\!\!-(q \wedge -\!\!\blacksquare\,\langle\circ\rangle\,\blacklozenge\!\!-p)$. So, pick x' such that $x\ R(-\!\!\blacksquare)\ x'$. Then there is a y' such that $x' \circ y'$ and $y' \models \blacklozenge\!\!-(q \wedge -\!\!\blacksquare\,\langle\circ\rangle\,\blacklozenge\!\!-p)$. Hence $y'\ R(\blacksquare\!\!-)\ y$. Furthermore, $y \models -\!\!\blacksquare\,\neg[\circ]\,\blacklozenge\!\!-p$, which means that for all $y_1\ R(\blacksquare\!\!-)\ y$ there exists an $x_1 \circ y_1$ such that $x_1\ R(\blacksquare\!\!-)\ x$. Now take $y_1\ R(\blacksquare\!\!-)\ y$. Let it be of depth δ_1. We can choose δ_1 such that $\gamma \leq \delta_1$. Now there exists a $x_1\ R(\blacksquare\!\!-)\ x$ such that $y_1 \circ x_1$. Now, $\circ$ is symmetric. By inductive hypthesis, therefore, y_1 and x_1 have the same depth. But y_1 has depth δ_1 and x_1 has depth $< \gamma$. Contradiction. Now assume conversely that x and y have the same depth. Pick any $x_1\ R(\blacksquare\!\!-)\ x$. It has depth $\delta < \gamma$, say. Then there exists a $y_1\ R(\blacksquare\!\!-)\ y$ of depth δ. By inductive hypothesis, $x_1 \circ y_1$. Analogously we can find x_1 for any given y_1. Since $\mathfrak{F} \models \nu$, therefore, putting $\beta(p) := \{x\}$ and $\beta(q) := \{y\}$ we find that $x \circ y$. This ends the proof. □

Now, observe the following. Let $<$ be a transitive order on R. Call a nonempty set $C \subseteq R$ **an inductive cone through** z if $z \in C$ and for all $y > z$: if $x \in C$ for all $x < y$ then also $y \in C$. C is an **inductive cone** if there is a z such that C in an inductive cone through z. An example of inductive cones are paths. Moreover, every inductive cone contains a path.

Lemma 5.6. *Put* $\mathrm{cf}(p) := p \wedge \blacksquare\!\!- (-\!\!\blacksquare\, p \to p)$. *Let $\mathfrak{F}$ be a Kripke–frame for* Π. *Then* $\langle \mathfrak{F}, \beta, x\rangle \models \mathrm{cf}(p)$ *iff $\beta(p)$ is an inductive cone through x.*

At last we add the following axiom.

Lemma 5.7. *Let $\mathfrak{F}$ be a Π–Kripke–frame and $\mathfrak{F} \models \nu$. Then $\mathfrak{F} \models \tau$ iff every branch of $\mathfrak{F}$ has the same order type. Here,*

$$\tau := \mathrm{cf}(p) \wedge \mathrm{cf}(q). \to .\blacksquare\!\!- (p \to \langle\circ\rangle q)$$

Proof. Suppose that every branch has the same order type and suppose that $\langle \mathfrak{F}, \beta, x\rangle \models \mathrm{cf}(p); \mathrm{cf}(q)$. Then $\beta(p)$ and $\beta(q)$ are inductive cones through x. Suppose that x_1 is such that $x\ R(\blacksquare\!\!-)\ x_1$ and $x_1 \in \beta(p)$. Then, as $\beta(q)$ contains at least one path and it has the same order type as any path through x_1, we see that there is a $y_1 \in \beta(q)$ of the same depth as x_1. Hence $x_1 \circ y_1$. It follows that $x_1 \models \langle\circ\rangle p$ and so $x \models \blacksquare\!\!- (p \to \langle\circ\rangle q)$. Hence $\mathfrak{F} \models \tau$. Assume now that $\mathfrak{F} \models \tau$. Let x be the root of $\mathfrak{F}$. Take two branches b and b' starting at x. These are inductive cones through x. Let them have well–order type γ and γ', respectively. Without loss of generality we may assume that $\gamma \geq \gamma'$. $\gamma = 1$ is a trivial case. So, let $\gamma > 1$. Put $\beta(p) := b$ and $\beta(q) := b'$. Now, $x \models \mathrm{cf}(p); \mathrm{cf}(q)$. Hence, $x \models \blacksquare\!\!- (p \to \langle\circ\rangle q)$. Take y of depth λ, $0 < \lambda < \gamma$ in b. Then $x\ R(\blacksquare\!\!-)\ y$, and so $y \models p$, from which $y \models \langle\circ\rangle p$. Hence there exists a y' such that $y \circ y'$ and $y' \models q$. So, y' is of depth λ and $y' \in \beta(q) = b'$. Hence $\gamma = \gamma'$. $\Box$

Definition 5.8. *Let* $\Pi^\ell := \Pi \oplus \nu \oplus \tau$.

Theorem 5.9. *Let $\mathfrak{F}$ be a Kripke–frame for Π^ℓ. Then $R(\blacksquare\!\!-)$ defines a homogeneous binary branching tree on F.*

The formula $\lambda := -\!\!\Box\bot$ is satisfiable exactly at the points whose depth is a limit ordinal. Now take the formula

$$\psi := \blacksquare\!\!-\ \blacklozenge\!\!-\lambda$$

A frame which satisfies ψ has the property that branches have depth at least ω^2. ψ is clearly consistent. Hence Π^ℓ has Kuznetsov–Index at least $\beth_1 = 2^{\aleph_0}$. Now, define Θ_1 to be the logic of all frames of Π^ℓ whose branches have countable depth.

Theorem 5.10. $Kz(\Theta_1) = 2^{\aleph_0}$.

So far we have not improved on our earlier example. Now we take a logic Λ. We first add an additional pair of operators, $\boxminus_W$ and $-\!\!\Box_W$, that define a well–order with end–points on the frames. The construction is as follows.

Definition 5.11. *Let* $\Lambda^{woe} := \Lambda^{wo} \oplus \boxminus_W \Diamond\!\!-_W \boxminus_W \bot$.

Lemma 5.12. *Let Λ be a κ–modal logic. Then for a rooted $\mathfrak{F} = \langle F, \langle \lhd_j : j < \kappa + 2\rangle\rangle$: $\mathfrak{F}$ is a Λ^{woe}–frame iff $\langle F, \langle \lhd_j : j < \kappa\rangle\rangle$ is a Λ–frame and $\lhd_\kappa$ is a well–order with end points.*

This follows from Lemma 3.7 using the fact that $\boxminus_W \Diamond\!\!-_W \boxminus_W \bot$ is true in a frame exactly when the well–order has an end point. That means that the well–ordering type is a successor ordinal. We note in passing that any set can be ordered using a well–order of such a type, so that if Λ is complete, Λ^{woe} is actually conservative over Λ.

Assume that the frame $\langle \{x\}, R\rangle$ with $R(\Box) = \varnothing$ (the one–point irreflexive frame) is a Λ–frame. Denote its logic by the letter Θ°. Form the logic Λ^+ by adding the modal operators for the binary trees, adding the postulates of Π^ℓ, and some axioms connecting the relations.

$$\begin{array}{rll}\Lambda^+ := & & \Pi^\ell \otimes \Lambda^{woe} \\ & \oplus & \blacksquare p \to \boxminus_W p \\ & \oplus & p \wedge \langle\circ\rangle\, \mathrm{n}(p) \to \boxminus_W \neg p \wedge \boxminus_W \neg p \\ & \oplus & \mathrm{n}(p) \wedge (\diamondminus_W \top \vee \diamondminus_W \top) \to [\circ](\neg p \to \boxminus_W \bot \wedge \boxminus_W \bot)\end{array}$$

Informally, the first postulate says that $R(\boxminus_W) \subseteq R(\blacksquare)$, the second that no two points of equal depth can be related via $R(\boxminus_W)$ and the third that there exists at most one branch along which the relation $R(\boxminus_W)$ is nontrivial. Hence, $R(\boxminus_W)$ is a disjoint sum of connected components, each of which is contained in a branch of $R(\blacksquare)$.

Lemma 5.13. *Let μ be an infinite cardinal number. Suppose that Λ is complete, $\Lambda \subseteq \Theta^\circ$. Then Λ^+ is conservative over Λ.*

Proof. Clearly, the reduct of a Λ^+–frame is a Λ–frame. So, it is enough if we show that each Λ–Kripke–frame is the reduct of some Λ^+–Kripke–frame. Consider a Λ–frame $\mathfrak{F} = \langle F, R\rangle$. We construct a Λ^+–frame as follows. First, we choose a well–ordering that makes $\mathfrak{F}$ into a Λ^{woe}–frame. This is possible. Assume therefore that $\mathfrak{F}$ already has this well–ordering, and that its type is γ. Now take a binary branching tree $\mathfrak{G} = \langle G, S\rangle$ in which every branch has order type γ. Select in $\mathfrak{G}$ a branch, b. There is a unique bijection $\xi : b \to F$ such that $\xi[S(\blacksquare) \cap b^2] = R(\boxminus_W)$, since also F has order type γ under $R(\boxminus_W)$. Now define S' as follows. For an operator $\Box$ of Π^ℓ put $S'(\Box) := S(\Box)$. Else put $S'(\Box) := \xi^{-1}[R(\Box)]$. Let $\mathfrak{H} := \langle G, S'\rangle$. We claim that $\mathfrak{H}$ is a Λ^+–frame. To that end, observe that the reduct of $\mathfrak{H}$ to the language of Π^ℓ is isomorphic to $\mathfrak{G}$ and the reduct to the language of Λ is isomorphic to a disjoint union of $\mathfrak{F}$ and some one–point irreflexive frames. Hence $\mathfrak{H} \models \Pi^\ell \otimes \Lambda$. Now, $S'(\boxminus_W) \subseteq S'(\blacksquare)$, since $x\ S'(\boxminus_W)\ y$ only if $x, y \in b$ and $x\ R(\blacksquare)\ y$. Furthermore, if $x\ S'(\boxminus_W)\ y$ then $x\ S'([\circ])\ y$ cannot hold, since then x and y have the same depth. So, $\mathfrak{H} \models \Lambda^+$. Finally, the relation is nontrivial along at most one branch. $\Box$

By Lemma 3.8, we deduce that if Λ is complete then $Kz(\Lambda^+) \geq Kz(\Lambda)$, and similarly for the modified Kuznetsov–Index. However, far better bounds can be obtained.

Lemma 5.14. *Let $\mathfrak{F}$ be a Λ^+–Kripke–frame and $\mathfrak{F}^-$ a rooted subframe of its reduct to Λ. If $\mathfrak{F}^-$ has cardinality $\mu \geq \aleph_0$ then $\mathfrak{F}$ has cardinality 2^μ.*

Proof. This follows from Proposition 3.1. $\Box$

Lemma 5.15. *Let Λ be a complete modal logic with Kuznetsov–Index$^{(\star)}$ μ. Then $\Theta^\circ \cap \Lambda$ is complete and has Kuznetsov–Index$^{(\star)}$ μ.*

This allows to show that the Kuznetsov–spectra are (almost) closed under exponentiation.

Lemma 5.16. *Let μ an infinite cardinal number. Suppose that there exists a modal logic with Kuznetsov–Index μ. Then there exists a modal logic with Kuznetsov–Index 2^μ.*

Proof. Let Λ be a logic with $Kz(\Lambda) = \mu$. We may assume that Λ is complete and $\Lambda \subseteq \Theta^\circ$. By Lemma 5.13, Λ^+ is conservative over Λ. Let Ξ be the logic of all Λ^+–Kripke–frames of cardinality $\leq 2^\mu$. By assumption on Λ, there is a formula φ which is μ–satisfiable but not λ–satisfiable for any $\lambda < \mu$. By Lemma 5.14, φ is 2^μ–satisfiable, but it is not κ–satisfiable for any $\kappa < 2^\mu$. Hence $Kz(\Xi) \geq 2^\mu$. By definition of Ξ, $Kz(\Xi) \leq 2^\mu$, and the claim is shown. $\Box$

Lemma 5.17. *Let μ be an infinite cardinal number. Suppose that there exists a modal logic with Kuznetsov–Index* μ. If $\mu = \lambda^+$, there exists a modal logic with Kuznetsov–Index* $(2^\lambda)^+$. Else, if $\mathrm{cf}(\mu) = \omega$, then there exists a modal logic with Kuznetsov–Index* $2^{<\mu} := sup\{2^\lambda : \lambda < \mu\}$.*

The proof is similar to the previous one. Notice that $sup\{2^\lambda : \lambda < \mu\} = sup\{(2^\lambda)^+ : \lambda < \mu\}$. (We remark that $\mu \leq 2^{<\mu} < 2^\mu$. This is about the only restriction on $2^{<\mu}$. The size of $2^{<\mu}$ otherwise depends very much on the universe.) We note the following consequences.

Corollary 5.18. $\mathrm{cf}(\rho_\alpha^f) = \mathrm{cf}(\rho_\alpha^{\star f}) = \omega$. $\omega \leq \mathrm{cf}(\rho_\alpha) = \mathrm{cf}(\rho_\alpha^\star) \leq 2^\omega$. *In particular, all Löwenheim numbers are singular.*

Corollary 5.19. $\rho_\alpha = \rho_\alpha^\star$. $\rho_\alpha^f = \rho_\alpha^{\star f}$.

Proof. We already know that $\rho_\alpha \leq \rho_\alpha^\star$. Now let $\mu \in \mathbb{K}_\alpha^\star$. Then if $\mu \notin \mathbb{K}_\alpha$ we have $\mu = \lambda^+$ with $\lambda \in \mathbb{K}_\alpha$. Now, $2^\lambda \in \mathbb{K}_\alpha$, by Lemma 5.16 and $\mu \leq 2^\lambda$. Since μ was arbitary, we have $\rho_\alpha^\star \leq \rho_\alpha$. The second claim is shown analogously. ☐

6. The Countable Limit

We have shown in the previous section how to create a logic with Kuznetsov–Index* 2^μ from a logic with Kuznetsov–Index* μ, on certain assumptions on μ. Here, we will deal with the countable limit of cardinal numbers. We will show a theorem both for μ and μ^+, where μ is a countable limit.

Lemma 6.1. *Suppose that μ is a cardinal number of cofinality ω. Suppose for a countable sequence $\langle \gamma_i : i \in \omega \rangle$ with limit μ there are complete logics Θ_i, such that $Kz^\star(\Theta_i) = \gamma_i$ and the one–point irreflexive frame is a Θ_i–frame. Then there is a logic Λ such that $Kz(\Lambda) = Kz^\star(\Lambda) = \mu$.*

Proof. All Θ_i are modal logics based on countable sets O_i of operators. We shall assume that the O_i are pairwise disjoint. Let $O := \bigcup_{i \in \omega} O_i$. Define $f : O \to \omega$ by $f(\Box) := i$ iff $\Box \in O_i$. Then form the logic

$$\Lambda := \bigotimes_{i \in \omega} \Theta_i \oplus \{\neg\Box\bot \to \Box'\bot : f(\Box) \neq f(\Box')\}$$

This is the fusion of all the Θ_i such that if there is a transition from x to some y in a frame using a Θ_i–modality then no transition from x to any point exists using a Θ_j–modality, where $j \neq i$. Now let $\mathfrak{F}$ be a Θ_i–frame. Extend $\mathfrak{F}$ to the frame $\mathfrak{F}^\circ$, in which $R^\circ(\Box) := R(\Box)$ if $f(\Box) = i$, and $R^\circ(\Box) := \varnothing$ if $f(\Box) \neq i$. Then $\mathfrak{F}^\circ$ is a Λ–frame. We call it a *simple extension.* It is easily established that Λ–frames are disjoint unions of simple extensions of some frames. Hence Λ is complete with respect to simple extensions. It follows that $Kz^\star(\Lambda) < \mu^+$ since any formula has a model based on a simple extension of a frame, and we can choose it to be less than γ_i in size. Now, for each $\delta < \mu$ there is an i such that $\delta < \gamma_i$. Furthermore, there is a formula φ consistent with Θ_i such that the least frame for φ has γ_i points. Now, the simple extension for that model is a Λ–model for φ. Moreover, any Λ–model for φ must have at least γ_i–many points, since it must contain a simple extension of a Θ_i–model for φ. This shows that Λ has Kuznetsov–Index* $\geq \mu$. Similarly it follows that the Kuznetsov–Index of Λ is $= \mu$. ☐

We note that if φ is a formula, one can actually construct a formula $\chi \vee \bigvee_{i<n} \psi_i$ such that χ is constant, ψ_i is in the language of Θ_i, $i < n$, and $\Lambda \vdash \varphi \leftrightarrow \bigvee_{i<n} \psi_i$. Namely, choose n large enough so that no modality of φ occurs in any of the Θ_i. Now choose modalities $\Box_i$, $i < n$, with $f(\Box_i) = i$.

$$\varphi = \varphi. \wedge . \bigwedge_{i<n} \Box_i\bot \vee \bigvee_{i<n} \Diamond_i\top$$

Now, $\varphi \wedge \bigwedge_{i<n} \bot$ can be reduced to a nonmodal formula in Λ, and $\varphi \wedge \Diamond_i\top$ can be reduced to a formula containing only modalities from Θ_i. This shows in detail why $Kz^\star(\Lambda) \leq \mu$.

Lemma 6.2. *Suppose that μ is a cardinal number of cofinality ω. Suppose for a countable sequence $\langle \gamma_i : i \in \omega \rangle$ with limit μ there are complete logics Θ_i, such that (a) a difference operator $[\neq_i]$ is definable in Θ_i for all $i < \omega$, (b) $Kz^\star(\Theta_i) = \gamma_i$, (c) the one–point irreflexive frame is a frame for Θ_i. Then there is a logic $\Lambda^\star$ such that $Kz(\Lambda^\star) = \mu$. $Kz^\star(\Lambda^\star) = \mu^+$.*

Proof. Proceed as in the previous example and define the logic Λ. Now extend Λ by two operators, $\boxplus$ and $\boxminus$, which are tense duals; moreover, $\boxplus$ satisfies $\mathsf{G.3}$, while $\boxminus$ satisfies

$$\boxplus\bot \wedge \boxminus\bot. \vee . \boxminus \neg \boxminus \bot$$

There are formulae φ_i such that φ_i can be satisfied in a Θ_i–frame of size at least γ_i, $i < \omega$. By Lemma 4.7 we may assume that they are without variables. Finally, for each $i < \omega$ add the postulates

$$\begin{array}{ll} \Diamond\top . \to . \boxplus^{i+1} \bot \wedge \neg \boxplus^{i} \bot, & \quad f(\Diamond) = i \\ \neg \boxplus \bot \to [\neq_i] \boxplus \bot & \\ \varphi_{i+1} \to \neg[\neq_{i+1}] \boxplus \neg\varphi_i & \\ \varphi_i \to \neg[\neq_i] \boxminus \neg\varphi_{i+1} & \end{array}$$

Here, $[\neq_i]$ is the difference operator of Θ_i. This defines the logic $\Lambda^\star$. Now, frames for $\Lambda^\star$ are made as follows. For each i, take a simple extension $\mathfrak{F}_i^\circ$ of a Θ_i–frame $\mathfrak{F}_i$. Let $\mathfrak{G} = \langle G, R\rangle$ be the disjoint union of these frames. $\mathfrak{G}$ is a frame for the reduct of $\Lambda^\star$ to the fragment without $\boxplus$ and $\boxminus$. $R(\boxplus)$ and $R(\boxminus)$ still need to be defined. We pick from each F_i, $i < \omega$, a point x_i. Now put $R(\boxplus) := \{\langle x_j, x_i\rangle : i < j\}$ and $R(\boxminus) := R(\boxplus)^\smile$. This completes the definition of $\langle G, R\rangle$. We claim that $\langle G, R\rangle \models \Lambda^\star$. This is obvious for the fragment without $\boxplus$ and $\boxminus$. (Note that we need condition (c) here to ensure that the disjoint union is a frame for Θ_i.) $R(\boxminus)$ is a disjoint union of well–order of type 1 or ω. Furthermore, there exists exactly one well–order of type ω, and in it the ith point is from $\mathfrak{F}_i^\circ$. Finally, the last two series of postulates say that if at the ith point of the well–order φ_i holds, then at the $i+1$st the formula φ_{i+1} holds. And if $i > 0$ then also at the $i-1$st point the formula φ_{i-1} holds. Now, consider the formula φ_0. It has a Θ_0–model of size γ_0. By construction, the only way to fulfill φ_0 is to create a disjoint sum of Θ_i–models $\langle \mathfrak{F}, \beta_i, x_i\rangle \models \varphi_i$, $i < \omega$, and defining $R(\boxplus) := \{\langle x_i, x_j\rangle : i > j\}$. [1] The resulting frame has cardinality $sup\{\gamma_i : i < \omega\} = \mu$. Moreover, by choice of the φ_i, no frame for φ_0 can have size $< \mu$. For then its size would be $< \gamma_j$ for some j. However, $\varphi_0 \vdash_{\Lambda^\star} \neg \boxplus \neg\varphi_j$, and no model for φ_j exists which has size $< \gamma_j$. Contradiction. So, the Kuznetsov–Index of $\Lambda^\star$ is at least μ and the Kuznetsov–Index* at least μ^+. Now, if $\Lambda^\star$ has Kuznetsov–Index $> \mu^+$, we may actually take the logic of the frame just presented, and we easily obtain a logic with Kuznetsov–Index μ^+. It is readily seen that this logic has Kuznetsov–Index μ. □

Theorem 6.3. *Let γ be a countable ordinal number. Then there exist logics Λ and $\Lambda^\star$ such that $Kz(\Lambda) = \beth_\gamma$ and $Kz^\star(\Lambda^\star) = \beth_\gamma$.*

Proof. We will show the result for the modified Kuznetsov–Index. We have seen that the result is true for $\gamma = 0$. In all examples presented, a difference operator is definable. The claim is true for each successor ordinal γ, the claim holds for $\gamma + 1$ if it holds for γ, by Theorem 5.17. Moreover, if there is a logic Λ_γ in which a difference operator is definable, then there is a logic $\Lambda_{\gamma+1}$ such that a difference operator is definable in it. (Namely, proceed from Λ to Λ^{wo} if necessary. This does not change the modified Kuznetsov–Index, by Lemma 3.9.) The cases where γ is a countable limit or a successor of a countable limit are covered by the previous results. □

Corollary 6.4. *(GCH.) Let γ be a countable ordinal number. Then there exist logics Λ and $\Lambda^\star$ such that $Kz(\Lambda) = \aleph_\gamma$ and $Kz^\star(\Lambda^\star) = \aleph_\gamma$.*

Corollary 6.5. *If $\alpha \geq \aleph_0$ then $\mathrm{cf}(\rho_\alpha) \geq \omega_1$.*

7. Simulating Countably Many Operators

In [13] it was described how modal logics with finitely many operators can be simulated by a single operator. This establishes already that for each n there is a monomodal logic with Kuznetsov–Index $\aleph_n$. However, if we want to reach higher, we need to simulate also countably many operators. This however is not as easy as in the finite case.

[1] We remark that the β_i are actually not needed, since the formulae are without variables. Moreover, notice that the frames $\mathfrak{F}_i$ need actually not be disjoint; the cardinality of the disjoint union is identical to the limit in either case.

Let $\mathfrak{F} = \langle F, R \rangle$ be a Kripke–frame based on $\aleph_0$ many operators, $\Box_i$, $i < \omega$. Then define a monomodal frame $\mathfrak{F}^s := \langle F^s, \lhd \rangle$, where

$$\begin{array}{rcl} F^s & := & (F \cup \{\star\}) \times \omega \\ \lhd & := & \left\{ \begin{array}{ll} & \{\langle\langle\star, i\rangle, \langle\star, j\rangle\rangle : i = j+1\} \\ \cup & \{\langle\langle x, i\rangle, \langle x, j\rangle\rangle : i \neq j, x \in F\} \\ \cup & \{\langle\langle x, i\rangle, \langle y, i\rangle\rangle : x, y, \in F, x \; R(\Box_i) \; y\} \end{array} \right. \end{array}$$

(We assume that $\star \notin F$.) We call a monomodal frame $\mathfrak{M}$ a **simulation frame** if it is of the form $\mathfrak{F}^s$ for some ω–modal Kripke–frame $\mathfrak{F}$. Given a complete ω–modal logic Λ we put $\Lambda^s := \mathsf{Th}\,(\mathsf{Krp}\,\Lambda)^s$. In other words, we take the logic of the frames simulating the Kripke–frames of Λ. The logic of all simulation frames, K^s_ω, is also called $\mathsf{Sim}(\omega)$. The following are theorems of this logic. (In contrast to the case of finitely many operators this set is not a complete set of axioms.)

$$\begin{array}{rcl} \omega_i & := & \Box^{i+1}\bot. \wedge .\neg\Box^i\bot \\ \alpha_i & := & \Diamond\omega_i. \wedge .\neg\omega_{i+1} \end{array}$$

$$\begin{array}{lll} (A) & \omega_i \wedge \Diamond p. \to .\Box p & \\ (B) & \alpha_i \to \Diamond\alpha_j, & i \neq j \\ (C) & \alpha_i \wedge \Diamond(\alpha_j \wedge p) \to \Box(\alpha_j \to p), & i \neq j \\ (D) & \alpha_i \wedge p \to \Box(\alpha_j \to \Diamond(\alpha_i \wedge p)) & i \neq j \\ (E) & \alpha_i \wedge \Diamond(\omega_i \wedge p) \to \Box(\omega_i \to p) & \\ (F) & \alpha_i \to \neg\Diamond\omega_j, & i \neq j \\ (G) & \Diamond^{\leq 3}(\omega_i \wedge p) \to \Box^{\leq 3}(\omega_i \to p) & \\ (H) & \omega_j \to \Diamond\omega_i & j > i \end{array}$$

Let the logic axiomatized by these postulates be Ψ. Clearly, $\Psi \subseteq \mathsf{Sim}(\omega)$. Now let $\mathfrak{M}$ be a Ψ–frame. Suppose that a point x satisfying some α_i is a root of $\mathfrak{M}$. We will show that although $\mathfrak{M}$ need not be a simulation frame, it does contain a subframe which is. Define $A_i := \{x : x \models \alpha_i\}$ and $\Omega_i := \{x : x \models \omega_i\}$. By (C), each point x in A_i sees exactly one point y in A_j, if $i \neq j$, and then by (D) we have $y \lhd x$. This establishes bijections $\psi_{ij} : A_i \to A_j$ such that for $x \in A_i$ and $y \in A_j$ we have $x \lhd y$ iff $y = \psi(x)$. Now, put $F := A_0$. Then a bijection ν from $F \times \omega$ to $\bigcup_i A_i$ is defined by $\nu(\langle x, i\rangle) := \psi_{0i}(x)$. Put now $R(\Box_i) := \nu^{-1}[\lhd \cap A_i^2]$. From (A) we see that each point in Ω_{i+1} has at most one successor in Ω_i. By (G) we see that in a rooted frame Ω_i contains exactly one point. Call it o_i. Extend ν by putting $\nu(\langle\star, i\rangle) := o_i$. By (H) and the definition of the ω_i, $o_j \lhd o_i$ iff $j > i$. By definition of the α_i, for every $x \in A_i$ we have $x \lhd o_i$, and by (F) we have $x \not\lhd o_j$ for $j \neq i$. We wish to claim that ν is a bijection. However this is not generally the case. Therefore define $S(\mathfrak{M}) := \bigcup_{i<\omega} A_i \cup \Omega_i$. Then $\mathfrak{M}$ induces on $S(\mathfrak{M})$ a frame which is isomorphic to a simulation frame. We put $\mathfrak{M}_s := \langle A_0, R \rangle$ with R defined above and call it the **unsimulation** of $\mathfrak{M}$.

We define for a formula in ω–many operators a simulation as follows.

$$\begin{array}{lcl} p^s & := & \alpha_0 \wedge p \\ (\neg\varphi)^s & := & \neg(\alpha_0 \wedge \varphi^s) \\ (\varphi \wedge \psi)^s & := & \varphi^s \wedge \psi^s \\ (\Box_i\varphi)^s & := & \Box(\alpha_i \to \Box(\alpha_i \to \Box(\alpha_0 \to \varphi^s))) \end{array}$$

Lemma 7.1. *Let $\mathfrak{N}$ be a Ψ–Kripke–frame. Suppose that $\langle \mathfrak{N}, \beta, x \rangle \models \alpha_0 \wedge \varphi^s$. Then there exists a valuation γ and a world y such that $\langle \mathfrak{N}_s, \gamma, y \rangle \models \varphi$.*

Proof. Define $F := A_0$, $R(\Box_i)$ as above and $\gamma(p) := \beta(p) \cap A_0$. Put $y := x$. It is shown by induction on φ that $\langle \mathfrak{F}, \gamma, y \rangle \models \varphi^s$. Namely, for variables we have $\langle \mathfrak{N}, \beta, x \rangle \models p^s$ iff $p \in \beta(p) \cap A_0$ iff $p \in \gamma(p)$ iff $\langle \mathfrak{F}, \gamma, x \rangle \models p$. The steps for $\neg$ and $\wedge$ are clear. The step for the modal operators is a straightforward calculation. □

Lemma 7.2. *Let $\mathfrak{F}$ be a $\mathsf{K}_{\aleph_0}$–Kripke–frame. Suppose that $\langle \mathfrak{F}, \gamma, y \rangle \models \varphi$. Let β be a valuation on $\mathfrak{F}^s$ such that $\beta(p) \cap F \times \{0\} = \gamma(p) \times \{0\}$. Then $\langle \mathfrak{F}^s, \beta, \langle y, 0\rangle \rangle \models \alpha_0 \wedge \varphi^s$.*

The proof is a straightforward induction on φ, which will be omitted. Now assume that Λ is a complete ω–modal logic with Kuznetsov–Index* μ, μ infinite. Look at the logic Λ^s. It is complete, by definition. Furthermore, it is complete with respect to Kripke–frames of size $< \mu \times \omega = \mu$. So Λ^s is μ–complete*. Let $\lambda < \mu$. Then there exists a formula φ_λ such that no Λ–Kripke–frame for φ_λ has size $< \lambda$. From Lemma 7.1 we see that if $\alpha_0 \wedge \varphi^s_\lambda$ has a Λ^s–Kripke–model based on $\mathfrak{N}$, then there is a model for φ_λ on its unsimulation $\mathfrak{N}_s$. By assumption, this frame has size $\geq \lambda$. Hence $\mathfrak{N}$ has size $\geq \lambda$. So, the Kuznetsov–Index* of Λ^s is $\geq \mu$.

Theorem 7.3. *For every $\aleph_0$–modal logic Θ, $Kz(\Theta) = Kz(\Theta^s)$ and $Kz^\star(\Theta) = Kz^\star(\Theta^s)$.*

Corollary 7.4. *Suppose that γ is a countable ordinal number. Then there exist monomodal logics $\Lambda^\star$ and Λ such that $Kz^\star(\Lambda^\star) = \beth_\gamma$ and $Kz(\Lambda) = \beth_\gamma$.*

Corollary 7.5. *(GCH.) Suppose that γ is a countable ordinal number. Then there exist monomodal logics $\Lambda^\star$ and Λ such that $Kz^\star(\Lambda^\star) = \aleph_\gamma$ and $Kz(\Lambda) = \aleph_\gamma$.*

We notice in passing the following. If Λ is a logic in which a universal modality is present then Λ^s is 3–transitive, that is, any point reachable from a given x is actually reachable in 3 steps. ($\mathsf{K4}$, by contrast, is 1–transitive.) So, we conclude that in the above theorem we can strengthen the assertion to Λ and $\Lambda^\star$ being 3–transitive.

As a result of these simulation theorems we note the following.

Theorem 7.6. *Let $0 < \alpha, \beta < \aleph_1$. Then*

1. $\mathbb{K}_\alpha = \mathbb{K}_\beta$ *and* $\rho_\alpha = \rho_\beta$.
2. $\mathbb{K}^\star_\alpha = \mathbb{K}^\star_\beta$ *and* $\rho^\star_\alpha = \rho^\star_\beta$.

In the light of this result we will now drop the index α and speak of $\mathbb{K}$, ρ and $\mathbb{K}^\star$ and $\rho^\star$.

However, notice that the spectra of finitely axiomatizable logics behave slightly differently. For if α is infinite, then $\mathbb{K}^{\star f}_\alpha = \{0\}$. Hence we only have the following, which is a consequence of the simulation results of [13] and the results of Section 4.

Theorem 7.7. *Let $0 < \alpha, \beta < \aleph_0$. Then*

1. $\mathbb{K}^f_\alpha = \mathbb{K}^f_\beta$ *and* $\rho_\alpha = \rho_\beta$.
2. $\mathbb{K}^{\star f}_\alpha = \mathbb{K}^{\star f}_\beta$ *and* $\rho^\star_\alpha = \rho^\star_\beta$.

We can draw from these results an interesting corollary.

Lemma 7.8. $\rho^f, \rho^{\star f} \in \mathbb{K} \cap \mathbb{K}^\star$.

Proof. Let Θ_i, $i < \omega$, be a enumeration of all monomodal logics which are finitely axiomatizable. By Lemma 6.1 there exists a logic Λ whose Kuznetsov–Index is the limit of all Kuznetsov–Indices of the Θ_i. By Corollary 7.4, there exists a monomodal logic with this property. Hence $\rho^f \in \mathbb{K}$. By Corollary 5.19, $\rho^f = \rho^{\star f}$. Finally, it is easily seen that $\rho^f \in \mathbb{K}^\star$ as well. ☐

Since $\rho_f \in \mathbb{K}$ and $\mathbb{K}$ has no maximal element we conclude the following

Theorem 7.9. $\rho_f < \rho$.

Furthermore, $\mathrm{cf}(\rho) \geq \omega_1$.

8. Reaching Higher

The methods so far can be improved rather drastically. Before we show how, we need to introduce some more tools. Recall from [13] the following characterization of modally definable first–order conditions. Let $\Box_j$, $j < \kappa$, be our basic operators. Define for a finite sequence $\vec{\sigma} \in \kappa^*$ the operator $\Box^{\vec{\sigma}}$ by induction.

$$\begin{array}{lcl} \Box^{\varepsilon}\varphi & := & \varphi \\ \Box^{i\vec{\sigma}}\varphi & := & \Box_i \Box^{\vec{\sigma}}\varphi \end{array}$$

Here, ε is the empty sequence. Furthermore, for a finite $s \subset \kappa^*$ put

$$\Box^s \varphi := \bigwedge_{\vec{\sigma} \in s} \Box^{\vec{\sigma}} \varphi$$

We may regard $\Box^{\vec{\sigma}}$ and $\Box^s$ actually as primitive operators, and it turns out that we have for any frame $\langle F, R \rangle$

$$\begin{array}{rcl} R(\Box^\varepsilon) & = & \{\langle x, x \rangle : x \in F\} \\ R(\Box^{i\vec{\sigma}}) & = & R(\Box^{\vec{\sigma}}) \circ R(\Box^i) \\ R(\Box^s) & = & \bigcup_{\vec{\sigma} \in s} R(\Box^{\vec{\sigma}}) \end{array}$$

A variable in a first–order formula is called **inherently universal** if it is quantified by a universal quantifier not in the scope of some existential quantifier. The following is shown in [13], Theorem 5.6.1.

Theorem 8.1. *$(\forall x)\alpha(x)$ is definable by means of a Sahlqvist formula iff it is equivalent to a formula that can be produced from constant formulae and formulae of the form $x\ R(\Box^s)\ y$ (called ground clauses) using $\wedge$ and $\vee$, and the restricted quantifiers $(\exists y)(x\ R(\Box^s)\ y \wedge \beta)$ and $(\forall y)(x\ R(\Box^s)\ y \to \beta)$ such that any ground clause contains at least one inherently universal variable.*

Now, in order to make use of this theorem, we observe the following. We know that with the introduction of a difference operator we also have the relation $\neq$. This allows to define the so–called **universal modality**, $[u]$, by

$$[u]\varphi := \varphi \wedge [\neq]\varphi$$

We have that $R([u]) = F \times F$ for any rooted Kripke–frame (the rootedness is necessary, of course). If we assume this, then we can actually define the unrestricted quantifiers; for if $\mathfrak{F}$ is rooted then $\mathfrak{F} \models (\exists y)(x\ R([u])\ y \wedge \beta(y))$ iff $\mathfrak{F} \models (\exists x)\beta(y)$.

Moreover, in [10] the so–called inaccessibility relation was introduced and axiomatized. Informally, if $\Box$ is any modal operator, then $\blacksquare$ is the corresponding **inaccessibility operator** or simply the **complement** of $\Box$ if $x\ R(\blacksquare)\ y$ iff not: $x\ R(\Box)\ y$. This can be put down with a simple axiom. Put

$$\mathrm{cm} := \langle u \rangle \mathrm{n}(p) \to (\Diamond p \leftrightarrow \neg \blacklozenge p)$$

Lemma 8.2. *A rooted Kripke–frame $\langle F, R \rangle$ satisfies* cm *iff $R(\blacksquare) = F^2 - R(\Box)$.*

This allows to lift the restrictions of the Sahlqvist theorem drastically.

Theorem 8.3. *Let α be a sentence in $R(\Box_j)$, $j < \kappa$, — possibly using restricted quantifiers —, such that every prime formula contains at least one inherently universal variable. Then the modal language can be enriched conservatively by some finitely many operators (and some axioms) such that α is definable by means of a Sahlqvist formula on all rooted Kripke–frames.*

Proof. First, we adjoin the difference operator by means of two relations. Next, for every negative ground clause $\neg(x\ R(\Box^s)\ y)$ we introduce the complement operator of $\Box^s$. Then, by appeal to Theorem 8.1, the theorem is proved: any negative ground formula can be replaced by a positive ground formula, and the unrestricted quantifiers are in fact restricted quantifiers (on rooted frames). $\Box$

A remark. The definition of cm is of course not Sahlqvist (otherwise the result would trivially follow from the earlier ones). We use this result to encode the axioms of set theory into modal logic. Even with the help of this theorem this turns out be a nontrivial exercise. For it is not simply guaranteed that all axioms of set theory are of the form required by the above theorem. Doing matters this way would also miss the point: there is a first–order axiomatization of set–theory, and if it were translated to modal logic the resulting logic admits small models, namely countable models. Hence, the trick is to use a mixture of first–order and second order axioms.

Let us start with the language in one operator, $[\in]$. We adjoin on the way some operators, always finitely many, in order to express our postulates. We illustrate the technique with some examples.

For ease of readability also we write $(\forall y \in x)\varphi$ and $(\exists y \in x)\varphi$ in place of $(\forall y)(y \in x \to \varphi)$ and $(\exists y)(y \in x \wedge \varphi)$, respectively.

- *Foundation.* There are no infinite descending $\in$–chains.
  Introduce the relation $\ni$ and its transitive closure $\ni^+$. Add the axiom G for $\ni^+$.
  $$[\ni^+]([\ni^+]p \to p) \to [\ni^+]p$$
  This ensures that no set contains an infinite descending $\in$–chain.
- *Extensionality.* $(\forall xy)(x \doteq y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y))$.
  This has the form required by the theorem. (For this formula is equivalent to
  $$(\forall xy)((x \doteq y \wedge (\forall z)(z \in x \leftrightarrow z \in y)) \vee (x \not\doteq y \wedge (\exists z)(z \in x \wedge z \notin y. \vee z \notin x \wedge z \in y)))$$
  In the first disjunct all variables are universally quantified, in the second z is existentially quantified. However, every prime formula contains either x or y, which are inherently universal.)
- *Set Union.* $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists u \in x)(z \in u))$.
  Introduce the relation symbol $\in^2$ together with the axiom
  $$(\forall xy)(x \in^2 y \leftrightarrow (\exists z)(x \in z \wedge z \in y))$$
  This satisfies the conditions of Theorem 8.3 and we may rewrite the first formula into
  $$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \in^2 x)\ .$$
  However, it still is not in the right form since the prime formula $z \in y$ contains no inherently universal variable. Therefore we adjoin a new relation U and some postulates such that $x\ U\ y$ iff $y = \bigcup x$. Since there is a unique union, the above postulate can in fact be rewritten into the required form. Namely, add the following axioms
  - $(\forall x)(\exists y)(x\ U\ y)$,
  - $(\forall xyz)(x\ U\ y \wedge x\ U\ z \to y \doteq z)$,
  - $(\forall xy)(x\ U\ y \leftrightarrow (\forall z)(z \in^2 x \leftrightarrow z \in y))$.

  Now the postulates are in the required form. The axiom is a consequence of these postulates.
- *Singleton Sets.* $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \doteq x)$.
  Adjoin a relation $\in_1$, with the intention that $x \in_1 y$ iff $y = \{x\}$. Now add the postulates
  - $(\forall x)(\exists y)(x \in_1 y)$,
  - $(\forall xyz)(x \in_1 y \wedge x \in_1 z \to y \doteq z)$,
  - $(\forall xy)(x \in_1 y \to x \in y)$,
  - $(\forall xyz)(y \in_1 x \wedge z \in x \to z \doteq y)$,
  - $(\forall xy)(y \in x \wedge (\forall z \in x)(z \doteq y) \to y \in_1 x)$.

  These postulates have the required form.
- *Powerset.* $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x)$.
  First, we define $\subseteq$. We have $x \subseteq y$ iff $(\forall z)(z \in x \to z \in y)$. Now, adjoin a relation $\subseteq$ (and an operator $[\subseteq]$) and the postulates
  - $(\forall xyz)(x \subseteq y \wedge z \in x \to z \in y)$,
  - $(\forall xy)(\exists z)(x \not\subseteq y \to z \in x \wedge z \notin y)$.

  After the introduction of the subset relation we introduce a relation P such that $x\ P\ y$ iff y is the powerset of x. The following postulates are added.
  - $(\forall x)(\exists y)(x\ P\ y)$,
  - $(\forall xyz)(x\ P\ y \wedge x\ P\ z \to y \doteq z)$,
  - $(\forall xy)(x\ P\ y \leftrightarrow (\forall z)(z \in y \leftrightarrow z \subseteq x))$.

  These postulates have the required form.

It already emerges that we can surround some problems by defining new relations, corresponding to set theoretic functions. This will become especially useful when talking about replacement. Further, if F is a relation corresponding to f (eg P is the relation corresponding to the powerset function $\wp$), we may also introduce the function f into our language. Locutions such as '$x = f(y)$' are equivalent to $y\ F\ x$, and so the syntactic description of Sahlqvist formulae remains intact even with functions. Moreover, one can also adjoin new unary predicates, which correspond to boolean constants. Here is a definition of $2(x)$, the property of having exactly two elements. It is mirrored by a boolean constant 2, with the following postulates:

1. $2 \to (\bigwedge_{i<3}\langle\ni\rangle p_i \to \bigvee_{i<j<3}\langle\ni\rangle(p_i \wedge p_j))$
2. $2 \to [\ni_1]\bot$

The first axiom says that if we have a node with property 2 then it has at most two successors, while the second says that no node is a singleton. Hence, $\langle\mathfrak{F}, \beta, x\rangle \models 2$ iff x has two elements iff $2(x)$. Incidentally, we also have $1(x)$, which is nothing but $(\exists y)(y \in_1 x)$. In general, we have the following theorem, which is easily derived from the Theorem 8.3.

Theorem 8.4. *Suppose that* cn *is a boolean constant symbol, and P is a unary predicate symbol. Let $Q(x)$ be a condition satisfying the conditions of Theorem 8.3. Then the condition $(\forall x)(P(x) \leftrightarrow Q(x))$ also has this property and there exists a Sahlqvist formula φ in some suitably enriched language, in which* cn *may occur, such that for any rooted Kripke–frame $\mathfrak{F}$, $\mathfrak{F} \models (\forall x)(P(x) \leftrightarrow Q(x))$ iff $\mathfrak{F} \models \varphi$.*

Analogously for modal operators $\Box$ and binary predicates Q, where the intended postulate is $x\ R(\Box)\ y \leftrightarrow Q(x, y)$. For replacement, we will have to define the notion of a *relation from x to y* and a *function from x to y*. First, we define therefore the notion of a pair p with components x and y. Recall that the pair $\langle x, y\rangle$ is defined as $\{x, \{x, y\}\}$.

- *Replacement.* p is a pair iff
  - p is a two element set $p = \{x, q\}$ such that $x \in q$,
  - $q = \{x\}$ or q is a two element set $q = \{x, y\}$. In the first case $\pi_1(p) := x$, $\pi_2(p) := x$ and in the second case $\pi_1(p) := x$ and $\pi_2(p) := y$.

  So, define
  - $(\forall x)(\mathrm{pair}_1(x) \leftrightarrow 2(x) \wedge (\exists y)(y \in x \wedge y \in^2 x \wedge (\forall z \neq y)(z \in x \wedge 1(z))))$,
  - $(\forall x)(\mathrm{pair}_2(x) \leftrightarrow 2(x) \wedge (\exists y)(y \in x \wedge y \in^2 x \wedge (\forall z \neq y)(z \in x \wedge 2(z))))$,
  - $(\forall x)(\mathrm{pair}(x) \leftrightarrow \mathrm{pair}_1(x) \vee \mathrm{pair}_2(x))$,
  - $(\forall xy)(x\ \pi_1\ y \leftrightarrow \mathrm{pair}(x) \wedge y \in x \wedge y \in^2 x)$,
  - $(\forall xy)(x\ \pi_2\ y \leftrightarrow (\mathrm{pair}_1(x) \wedge x\ \pi_1\ y) \vee (\mathrm{pair}_2(x) \wedge y \in^2 x \wedge \neg(y \in x)))$.

  We introduce relations η_1 and η_2 together with the axioms
  - $(\forall yz)(y\ \eta_1\ z \leftrightarrow \pi_1(y) = \pi_1(z))$,
  - $(\forall yz)(y\ \eta_2\ z \leftrightarrow \pi_2(y) = \pi_2(z))$.

  Next, we introduce unary predicates rel and fun with the following definitions
  - $(\forall x)(\mathrm{rel}(x) \leftrightarrow (\forall y)(y \in x \to \mathrm{pair}(y)))$,
  - $(\forall x)(\mathrm{fun}(x) \leftrightarrow \mathrm{rel}(x) \wedge (\forall y)(\forall z\ \eta_1\ y)(z\ \eta_2\ y))$.

  Finally, we introduce the relations dom and rng. They are partial functions, denoted by the same symbols. We abbreviate by $f(x)\downarrow$ the fact that f is not defined on x and by $f(x)\uparrow$ the fact that f is defined on x. (These are equivalent to the formulae $\neg(\exists y)(y \doteq f(x))$ and $(\exists y)(y \doteq f(x))$, respectively.
  - $(\forall xyz)(x\ \mathrm{dom}\ y \wedge x\ \mathrm{dom}\ z \to y \doteq z)$,
  - $(\forall xy)(y \in \mathrm{dom}(x) \leftrightarrow (\exists z)(z \in x \wedge y \doteq \pi_1(z))) \vee \mathrm{dom}(x)\downarrow$,
  - $(\forall xyz)(x\ \mathrm{rng}\ y \wedge x\ \mathrm{rng}\ z \to y \doteq z)$,
  - $(\forall xy)(y \in \mathrm{dom}(x) \leftrightarrow (\exists z)(z \in x \wedge y \doteq \pi_1(z))) \vee \mathrm{dom}(x)\downarrow$.

  The axiom of replacement becomes

$$(\forall xy)(\mathrm{fun}(x) \wedge y \doteq \mathrm{dom}(x) \to (\exists z)(z \doteq \mathrm{rng}(x)))\ .$$

Finally, we turn to the axiom of comprehension. Unlike in first–order theories we do not require that from a given set we single out those elements that satisfy a given property. Rather, our axiom says something like this. If x is a set (that is, a point in the Kripke–frame) and we have a collection Y of points then there is a set y that contains exactly those sets which are in x and in Y. By replacing Y by $Y \cap x$ we can derive the (equivalent) condition:

$$(\forall Y)(\forall x)(Y \subseteq x \to (\exists y)(\forall z)(z \in y \leftrightarrow z \in Y))\ .$$

Hence, we are playing with the sets of the metatheory (called 'collections' or 'classes') and the sets of the model itself. The axiom is the following. (Here, $\overline{\ni}$ is the complement of $\ni$.)

- *Comprehension.* $\langle u\rangle \mathrm{n}(q) \wedge [u](p \to [\in]q) \to \langle u\rangle([\ni]p \wedge [\overline{\ni}]\neg p)$

A Kripke–frame satisfies this formula iff for all collections $Y = \beta(p)$ and $Z = \beta(q)$: if Z is a set (!) and every member of Y is $\in$–related to Z (in other words, if $Y \subseteq Z$), then there is a set v such that all members of u are in Y and no member is not in Y. In other words, $v = Y$, and so Y is a set. This means that every subcollection of a set is a set.

Now, several auxiliary notions can be defined. x has the same cardinality as y — in symbols $x \sim y$ — iff there exists a bijective function from x to y. For simplicity, we make use of the Cantor–Bernstein–Theorem. We first define 'x is of lesser cardinality than y', $x \leq y$, and then define $x \sim y$ by $x \leq y \wedge y \leq x$.

1. $(\forall x)(\mathrm{inj}(x) \leftrightarrow \mathrm{fun}(x) \wedge (\forall y)(\forall z\ \eta_2\ y)(y\ \eta_1\ z))$,
2. $(\forall xy)(x \leq y \leftrightarrow (\exists z)(\mathrm{inj}(z) \wedge \mathrm{dom}(z) \doteq x \wedge \mathrm{rng}(z) \doteq y)))$,
3. $(\forall xy)(x \sim y \leftrightarrow x \leq y \wedge y \leq x)$.

Readers may have noted that unary predicates sometimes occur but the variable is not inherently universal. Since unary predicates correspond to boolean constants, and the occurrences of constants are not restricted by the Sahlqvist Theorem (see [13]), it follows that there is no restriction on occurrences of prime formulae using unary predicates. A quick way to see this is as follows. If P is a unary predicate, introduce a binary predicate Q with the axiom $(\forall xy)(Q(x,y) \leftrightarrow P(y))$. This is Sahlqvist. Now let α be a formula with occurrences of P. Replace occurrences of $P(y)$ by $Q(x,y)$, where x is inherently universal. Call the result α^Q. Then if all binary relation symbols of α satisfy the conditions, so does α^Q. If this is done for all unary predicates, we end up with a formula that is Sahlqvist. So, there are no conditions on unary predicates.

An *ordinal* is a set which is transitively and linearly ordered by $\in$ (that it is also well–ordered by $\in$ follows from the foundation axiom). To define this property we introduce the relation $\heartsuit$ defined by

$$(\forall xy)(x \heartsuit y \leftrightarrow x \notin y \wedge x \neq y \wedge y \notin x)$$

($x \heartsuit y$ iff x and y are (different and) $\in$–incomparable.) Therewith we define a property $\mathrm{ord}(x)$ by

$$(\forall x)(\mathrm{ord}(x). \leftrightarrow .(\forall y)(y \in^2 x \to y \in x) \wedge (\forall y \in x)(\forall z \heartsuit y)(z \notin x))$$

This can defined in modal terms, by Theorem 8.4. Using the ordinals we can install the axiom of infinity in the following way: we define 'limit ordinal' by

$$(\forall x)(\mathrm{lord}(x) \leftrightarrow (\exists y)(y \in x) \wedge (\forall y)(y \in x \to y \in^2 x))$$

- *Infinity.* $(\exists x)\mathrm{lord}(x)$.

A cardinal number is an ordinal y such that for every ordinal x: if $x < y$ then $x \sim y$ does not hold. Again, using Theorem 8.4 this can be rendered into modal terms.

- *Choice.* The axiom of choice is equivalent to the axiom of well–ordering. Hence we take as axiom the statement

$$(\forall x)(\exists y)(\mathrm{ord}(y) \wedge x \sim y)$$

  This has the required form.

In this way, all axioms of set theory ZFC are converted into modal axioms involving some expansion of the original signature by some finite set of operators. Call the resulting logic Σ. Let Σ^- be the logic without the axiom of replacement. We do not know whether Σ^- or Σ are complete.

Lemma 8.5. *Suppose that $\mathfrak{F}$ is a rooted Kripke–frame for Σ. Then $\langle F, \in\rangle$ satisfies the axioms of ZFC. Moreover, every class contained in a set is a set.*

Likewise for Σ^-. We define the restricted universes V_λ, λ an ordinal, in the usual way:

$$\begin{array}{lcll} V_0 & := & \{\varnothing\} & \\ V_{\lambda+1} & := & \wp(V_\lambda) & \\ V_\lambda & := & \bigcup_{\mu<\lambda} V_\mu & \lambda \text{ limit ordinal} \end{array}$$

Here $\varnothing$ is the unique $\in$–minimal member of V.

Lemma 8.6. *Let $\mathfrak{F}$ be a rooted Kripke–frame for Σ^-. Then $|V_{\lambda+1}| = 2^{|V_\lambda|}$. It follows that $|V_{\omega+\lambda}| = \beth_\lambda$.*

Proof. It is enough to observe that $V_{\lambda+1}$ is in one to one correspondence with the classes of V_λ. Hence its cardinality is 2^μ, where μ is the cardinality of V_λ. ☐

Theorem 8.7. *A rooted Kripke–frame of Σ^- has cardinality $\beth_\lambda$, λ a limit ordinal.*

A cardinal μ is **inaccessible** if it is $> \aleph_0$, regular, and a strong limit. μ is **regular** if it is not the supremum of $< \mu$ many cardinals, and a **strong limit** if $2^\nu < \mu$ for every $\nu < \mu$. (See Jech [11].)

Theorem 8.8. *The Kuznetsov–Index of Σ is either 0 or some inaccessible cardinal. It is 0 iff there exists no inaccessible cardinal.*

Proof. Let $\langle F, R\rangle$ be a Kripke–frame for Σ. Then $\langle F, R([\in])\rangle$ is a model of ZFC set theory. It follows that $|F|$ must be an inaccessible cardinal. If inaccessible cardinals do not exist, then $F = \varnothing$ and so the Kuznetsov–Index of Σ is 0. Otherwise, let α be inaccessible. Then $\langle V_\alpha, \in\rangle$ is a model of ZFC. It can be turned into a frame for ZFC by interpreting $\in$ as the relation $R([\in])$ and suitably defining $R(\Box)$ for the other operators. ☐

It is not hard to see that the logic of the smallest model of ZFC in the signature of Σ is such that its Kuznetsov–Index is 0 if no inaccessible cardinal exists and that it is the smallest inaccessible cardinal otherwise. Notice that the consistency of Σ is independent of the existence of inaccessible cardinals, since it is only a finitary notion. It follows that if Σ is consistent but no inaccessible cardinals exist, then Σ has no Kripke–frames and is therefore incomplete. Hence, the completeness of Σ depends on the structure of the universe.

9. Some facts about ρ

Let us recall the facts so far. $\mathbb{K}_\alpha$ does not depend on α, and so ρ_α is independent of α as well. Furthermore, $\rho_\alpha = \rho_\alpha^\star$. $\mathbb{K}$ is a set of cardinality $\leq 2^{\aleph_0}$, and it is closed under countable limits and $\mu \mapsto 2^\mu$. Now, what is the size of ρ? We will establish here a charcterization in terms of definability. The results obtained here make heavy use of certain set theoretic constructions, which are explained in detail in [7].

To approach this question, we will compare the expressive strength of modal logic with that of monadic second order logic. $\mathfrak{L}$ is a language of monadic second order logic (MSO) if it contains

- a countable set of individual variables and a countable set of class variables,
- enough boolean connectives (eg $\top$, $\wedge$ und $\neg$)
- the first–order quantifiers $\forall$ and $\exists$,
- the second–order quantifiers $\forall$ and $\exists$,
- at most countably many first–order constants, functions and relations,
- at most countably many class constants.

Pure MSO is that particular language that has no first–order constants, functions or relations, except equality, and has a single class constant, denoted by $\underline{U}$. A Π^1_1–formula is a formula of MSO in which the class variables are only universally quantified. A Σ^1_1–formula is a formula in MSO in which the class variables are only existentially quantified over.

The standard translation, $\varphi^\dagger$, defined in Section 4 defines a translation of modal logic into first order logic. We let φ^δ be defined by

$$\begin{array}{lcl} p_i^\delta & := & P_i(x) \\ (\neg\varphi)^\delta & := & \neg\varphi^\delta \\ (\varphi\wedge\psi)^\delta & := & \varphi^\delta \wedge \psi^\delta \\ (\Box\varphi)^\delta & := & (\forall y)(x \; R(\Box) \; y \to \varphi^\delta[y/x]) \end{array}$$

Now, let $(\forall\vec{P})\varphi^\delta$ be the universal closure of φ^δ. So, $\vec{P} = P_0 \dots P_{n-1}$, where all occurring variables of φ are of the form p_i, $i < n$. Then we have

$$\mathfrak{F} \models \varphi \quad \text{iff} \quad \mathfrak{F} \models (\forall\vec{P})\varphi^\delta$$

Consequently, a modal logic defines a set of structures that is definable by a set of Π^1_1–sentences.

If we read $P_i(x)$ simply by $x \in P_i$, $(\forall\vec{P})\varphi^\delta$ is a Π^1_1–sentence in the language of set theory. We wish to show now that conversely for any Π^1_1–sentence ψ there exists a modal formula $\psi^\ddagger$ such that $\mathfrak{F} \models \psi$ iff $\mathfrak{F} \models \psi^\ddagger$, given that we may actually enrich the signature somewhat. This will be enough to show that the number ρ can be equated with an analogously defined number for a set of Π^1_1–formulae. There are two ways to proceed. The first is interesting in its own right but will not lead to a full result. Only the second method succeeds.

Here is the first method. Recall the Theorem 8.1. Using the methods of [12] one can actually lift this theorem to Π^1_1–sentences of the form $(\forall\vec{P})(\forall x)\alpha(\vec{P}, x)$, where ground clauses are of the form $(\neg) y\, R(\Box_j)\, y'$ or $y \in P_i$ or $y \notin P_i$. There is no condition on the variable y in the last two cases. In particular, it need not be inherently universal. Going through the same arguments of the previous section one can then show that any sentence $(\forall\vec{P})(\forall x)\alpha(\vec{P}, x)$ is modally definable on rooted Kripke–frames in an enriched signature if only any ground clause of the form $y\, R(\Box_j)\, y'$ or its negation contains at least one inherently universal variables.

We will now show that the last condition can almost be eliminated if we are working in the language of set–theory. First, we can reduce the second–order prefix to a single variable, using the typical coding of sequences by sets. Further, assume that the formula is not second order but first–order. We then introduce Skolem–functions to eliminate all existentially quantified variables. For example, $(\forall\vec{x})(\exists y)\varphi(\vec{x}, y)$ becomes

$$(\forall\vec{x})\varphi(\vec{x}, f(\vec{x}))$$

and the additional postulates ensuring that f is a function are clearly special. However, Skolem–functions are not necessarily unary. So, we replace an n–ary Skolem–function f by the function $f^\heartsuit$, defined on n–tuples of sets. If π^n_i denotes the projection of an n–tuple to its ith coordinate, we require therefore that

$$f^\heartsuit(y) = f(\pi^n_0(y), \pi^n_1(y), \dots, \pi^n_{n-1}(y)) .$$

We introduce into the formula $(\forall\vec{x})(\exists y)\varphi(\vec{x}, y)$ the function $f^\heartsuit$ rather than f. Hence we have to transform the formula into

$$(\forall\vec{x})(\forall y)(\bigwedge_{i<n} \pi^n_i(y) \doteq x_i \to \varphi(\vec{x}, f^\heartsuit(\vec{x})))$$

It is readily checked that the formula expressing that $f^\heartsuit$ is defined only on n–tuples is special. So, we have replaced the existential quantifier by a universal quantifier at the price of introducing only a binary function.

If the formula is however not first–order but truly second order, matters are not so easy. For then the Skolem–function, in addition to depending on the first–order variables may also depend on the second–order variable(s). Let us therefore try another method. Consider any signature of MSO. Recall that we may have countably many first–order relation and function symbols (and constants). We can however recode the relation and function symbols by means of a single class U which describes them. (To see how, note that we may write countably many relations on V as a single subset of V^ω, which again can be recoded into V. All these codings are elementarily definable.) Therefore, we add some constant $\underline{U}$ to denote this class and translate a formula φ into $\varphi^\spadesuit$, which is a formula of pure MSO with one constant, $\underline{U}$, and one relation symbol, $\in$ in addition to equality. So, any Π^1_1–sentence φ of the original language is satisfiable in a second order model expanding the universe $\langle V, \in\rangle$ by relations and functions iff there exists some U such that $\langle V, \in, U\rangle$ satisfies $\varphi^\spadesuit$. Furthermore, if φ is Π^1_1, so is $\varphi^\spadesuit$. Since we can use boolean constants to denote classes, $\varphi^\spadesuit$ is by the results established above a sentence that is modally definable in a suitable signature!

Definition 9.1. *κ is the **index** of some countable set T of Π^1_1–formulae if the smallest model for T has cardinality κ. If T is finite, κ is called a **finitary index**. Let $\mathbb{P}$ be the set of indices and $\mathbb{P}^f$ the set of finitary indices. Finally, put $\pi := sup\,\mathbb{P}$ and $\pi^f := sup\,\mathbb{P}^f$.*

Theorem 9.2. *Suppose that $\mu = \beth_\alpha$ for some α which is 0 or a limit ordinal. Then $\mu \in \mathbb{K}$ iff $\mu \in \mathbb{P}$ and $\mu \in \mathbb{K}^f$ iff $\mu \in \mathbb{P}^f$.*

Proof. (Sketch.) Observe that for the reduction of Π^1_1–formulae into modal logic we do not need full set–theory but rather enough so that we can code countable sequence of sets by sets. So, the reduction works actually in ZFC minus Replacement. Models for this theory can be built on V_α for any limit ordinal α. So, $\alpha = \omega + \beta$ for some β such that $\beta = 0$ or β a limit ordinal. Now notice that $|V_\alpha| = \beth_\beta$. □

Corollary 9.3. *$\rho = \pi$ and $\rho^f = \pi^f$.*

This shows that as far as the number ρ is concerned we may actually work in MSO instead.

We will close our investigations with some remarks concerning the omission of certain cardinals. Recall the notion of an *indescribable cardinal.* A cardinal α is Π^1_1–**indescribable** if for every Π^1_1–sentence φ of pure MSO, if $\langle V_\alpha, \in, U\rangle \models \varphi$ then for some $\beta < \alpha$: $\langle V_\beta, \in, U \cap V_\beta\rangle \models \varphi$. The first to note is that this notion of indescribability can be extended to countable sets of sentences.

Lemma 9.4. *Suppose that α is Π^1_1–indescribable. Let Φ be a countable collection of sentences in pure MSO. Then if $\langle V_\alpha, \in, U\rangle \models \Phi$, there exists a $\beta < \alpha$ such that $\langle V_\beta, \in, U \cap V_\beta\rangle \models \Phi$.*

Proof. We can code formulae in set–theory by means of so–called *Gödel–sets.* These are hereditarily finite sets, hence members of V_ω. In particular, note that (1) the predicate $G(x)$, defining the set of Gödel sets, is elementary, (2) each Gödel–set is elementarily definable. Now, there exists a formula $\chi(x)$ in which only x occurrs free and which is universal for Π^1_1. This means that for all Π^1_1–sentences φ, all limit ordinals $\alpha > \omega$ and all $U \subseteq V_\alpha$:

$$\langle V_\alpha, \in, U\rangle \models \varphi \leftrightarrow \chi(u_\varphi)$$

where u_φ is the Gödel–set corresponding to φ. Now consider the set $G(\Phi) := \{u_\varphi : \varphi \in \Phi\}$. This is a subset of V_ω. Furthermore,

$$\langle V_\alpha, \in, U\rangle \models \Phi \text{ iff } \langle V_\alpha, \in, U\rangle \models (\forall x)(x \in G(\Phi) \to \chi(x))$$

Now add a constant $\underline{P}$ to the language, which may be interpreted by any class. Then there is a $P \subseteq V_\alpha$ such that

$$\langle V_\alpha, \in, U, P\rangle \models (\forall x)(G(x) \wedge \underline{P}(x) \to \chi(x))$$

iff $\langle V_\alpha, \in, U\rangle \models \Phi$. We may recode U and P into a single class and call it U again. For example, we may do this in such a way that the finite sets of U are exactly the Gödel–sets of Φ. After this recoding we have

$$\langle V_\alpha, \in, U\rangle \models (\forall x)(G(x) \wedge \underline{U}(x) \to \chi(x))$$

Now if α is Π^1_1–indescribable there exists a $\beta < \alpha$ such that

$$\langle V_\beta, \in, U \cap V_\beta\rangle \models (\forall x)(G(x) \wedge \underline{U}(x) \to \chi(x))$$

Now $V_\beta \cap V_\omega = V_\alpha \cap V_\omega$ (remember we have at least ZF–set theory, so the levels are identical and α and β are limit ordinals and $> \omega$). It follows by the universality of χ that

$$\langle V_\beta, \in, U \cap V_\beta\rangle \models \Phi$$

This shows the claim. □

Now, we may also speak of a *modally indescribable cardinal*, which is a cardinal such that whenever for a modal logic Θ containing Σ and $\langle V_\alpha, \in, R\rangle \models \Theta$ there exists a $\beta < \alpha$ such that $\langle V_\beta, \in, R \restriction V^2_\beta\rangle \models \Theta$. (Here, $R \restriction V^2_\beta$ is the function returning for each j the set $R(\Box_j) \cap V^2_\beta$.) It is clear that a modally indescribable cardinal does not belong to $\mathbb{K}$. Further, by the Lemma 9.4 and the reduction of Θ to a countable set of Π^1_1–sentences and vice versa we establish the

Theorem 9.5. *A cardinal is modally indescribable iff it is Π^1_1–indescribable.*

It is easy to see that in particular ρ is Π^1_1–indescribable, which means that it cannot be defined without parameters by a single Π^1_1–formula, nor, as we have seen, by a countable set of such formulae.

10. Conclusion

Some implications of the previous results shall be mentioned. Suppose that we have a Kripke–frame $\mathfrak{F}$ for Σ inside some universe V. Then, by the fact that we have second order set comprehension, one can show that $W := \langle F, R(\in)\rangle$ is isomorphic to $\langle V_\alpha, \in\rangle$ for some ordinal α. We shall now identify objects of W modulo this isomorphism with objects of V_α. Then we get the following facts. Given two objects, x and y of V_α, we have $W \models$ "$|x| = |y|$" iff $V \models$ "$|x| = |y|$". In other words, the notion of cardinality does not depend on whether we look at it from inside the model or from outside. This is meant when one says that *having the same cardinality* is *absolute in* W. Similarly for the notion of a cardinal. So, we have $W \models$ "x is a cardinal" iff $V \models$ "x is a cardinal". We also say that x is a cardinal^W to say that in $W \models$ "x is a cardinal". Notice that the notion of an ordinal is elementarily definable inside a ZFC–model and so also absolute. (The notion of a well–order is Π^1_1–definable.) Just a little reflection on the comprehension axiom shows that the notions of powerset, of a product of two sets, a relation between two sets etc are the same in the model as in the universe V. So, explicit set–theoretic constructions do not depend on whether we perform them outside or inside W. As a consequence we get that x is inaccessible^W iff it is inaccessible^V. In sequel we shall take V to be the total universe and we shall drop the superscript V.

Now, a cardinal number is **weakly compact** iff it is inaccessible and has the tree property: a cardinal μ has the **tree property** iff

> for every tree T on μ of order μ such that for each $\lambda < \mu$ fewer than μ elements have order λ then T has a branch of order μ.

(See [5].) Here, a tree is a pair $\langle T, <\rangle$ such that $<$ satisfies certain axioms and such that for all y the set $\{x : x < y\}$ is well–ordered by $<$. The well–order type of this set is called the **order** of y. The order of the tree is the supremum of all orders of its elements. Now we claim the following:

Lemma 10.1. $W \models$ "μ has the tree property" *iff μ has the tree property.*

Proof. Suppose that $W \not\models$ "μ has the tree property". Then there is a tree $\langle \mu, <\rangle$ (in W) such that for each $\lambda < \mu$ there are fewer^W than μ elements of order λ but $\langle \mu, <\rangle$ does not have an element of order μ. Since x has fewer^W elements than y iff x has fewer elements than y, μ fails to have the tree property. Conversely, let μ fail to have the tree property. Then there is a tree $T = \langle \mu, <\rangle$ exemplifying this. Now, $<$ is a subset of $\mu \times \mu$, and so, by Π^1_1–comprehension, $<$ is a set in W. Likewise it can be shown that T is a set in W, and so $W \not\models$ "μ has the tree property". □

Hence, consider the first–order axiom $\tau(x)$ stating that x has the tree property^W and $\mathrm{inacc}(x)$ that x is inaccessible^W. Then we have seen that $W \models \exists x.\tau(x) \wedge \mathrm{inacc}(x)$ iff W contains a weakly compact cardinal. In a similar vein we can write down a first–order statement $\mathrm{ms}(x)$ such that $\mathrm{ms}(x)$ is true iff x is measurable^W. It can be proved that x is measurable^W iff it is measurable, and this will demonstrate that if measurable cardinals exists, ρ is greater than the least measurable cardinal! This shows that the Löwenheim number of modal logic, even though it can be shown to exist, in general exceeds any large cardinal that can be defined by means of a higher order sentence (if that cardinal exists), since higher order quantification is reducible to first–order quantification in presence of full comprehension, as long as we quantify over classes that are bounded by some definable set–theoretical function of the occurring (first–order) set–variables. This is the case with quantifying over trees over a cardinal or over ultrafilters on a cardinal κ, which are subsets of $\wp^n(\kappa)$ for some suitable n.

Let us briefly mention that although we have succeeded in characterizing ρ, the identity of ρ^f remains unclear. For the logics we have defined above are finitely axiomatizable, but we have not shown them to be complete. Since there always is a completion, this was enough for establishing a lower bound for ρ. However, it is not in general the case that the completion of a finitely axiomatizable logis is finitely axiomatizable again. So we lack an essential link here to establish lower bounds for ρ^f. Notice by the way that the completeness of Σ may well depend on the size of the universe, though its consistency is independent of it. Finally, the Löwenheim numbers of $\mathsf{K4}$ logics are also not known.

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