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F. Wolter, H. Wansing, M. de Rijke, and M. Zakharyaschev, eds

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The Semantics of Modal Predicate Logic. Part 1: Completeness

MARCUS KRACHT AND OLIVER KUTZ

ABSTRACT. We introduce a new semantics for modal predicate logic, with respect to which a rich class of first–order modal logics is complete, namely all normal first–order modal logics that are extensions of free quantified **K**. This logic is defined by combining positive free logic with equality $\mathbf{PFL}^{\doteq}$ and the propositional modal logic **K**. We then uniformly construct—for each modal predicate logic $\mathcal{L}$—a canonical model whose theory is exactly $\mathcal{L}$. This proves completeness with respect to so–called modal–structures. We add some remarks on canonicity and frame–completeness and finally show that if suitable modal algebras of 'admissible interpretations' are added to modal predicate frames, general frame–completeness is gained.

1 Introduction

In the past several years there has been a continuous interest in so–called generalized Kripke–semantics or Kripke–type–semantics. The main motivation for this revival of interest in the semantics for first–order modal logics basically comes from two different sources. First of all, it has been realized that standard Kripke–semantics is highly incomplete as witnessed e.g. by a theorem of Ghilardis (cf. Ghilardi 1991) . This has led to several generalizations—such as functor semantics, Kripke–sheaves or metaframe–semantics—which were on the one hand mathematically rich enough to allow for general completeness results and on the other hand enabled one to understand the insufficiency of standard Kripke–semantics much better (cf. Skvortsov and Shehtman 1993) .

Secondly, there has been a growing awareness that standard seman-

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tics is also inadequate from a more philosophical point of view or that standard syntactic machinery is insufficient with respect to a proper treatment of e.g. singular terms or definite descriptions in modal contexts (cf. Fitting and Mendelsohn 1998) . One should also mention here different attempts to modify or generalize the Counterpart–Theory by David Lewis.

The main problem obviously lies in the question of how to treat “modal individuals” . Observe that in standard semantic approaches like constant– or varying domain semantics, variables and constants are treated quite differently. While the former take *objects* as values that are supposed to be—once fixed—constant throughout all possible worlds (apart from changing all their properties) the latter can very well take *individual concepts* as values as is the case with non–rigid constants. This is hence the starting point for generalizations of the semantics. Note, that there is also an alternative way of dealing with this asymmetry, i.e. to explicitly allow quantification over both individual objects and individual concepts as has been done in (Fitting 2000) .

Now, there is still no consensus as to which generalized semantics is the ‘right’ one from a purely mathematical point of view. Furthermore, the high technicality of the proposed semantics has led to—as a matter of fact—almost complete neglect on behalf of philosophers, linguists etc., despite their far greater flexibility in handling e.g. problems of the individuation of individuals.

In this paper we first single out a rich class of first–order modal logics, which are normal extensions of our base logic **FK**. This base logic is obtained from a combination of positive free logic with identity, $\mathbf{PFL}^{\doteq}$, and the modal propositional logic **K**. We then present a semantics which is on the one hand closely related to the functor–semantics proposed by Ghilardi (cf. e.g. Ghilardi 1991) and on the other hand picks up some ideas from David Lewis’ Counterpart–Theory (Lewis 1968) to deal e.g. with the failure of the principle of the necessity of identity. We follow a proposal by Van Benthem and avoid all categorial terminology that is not strictly necessary (cf. van Benthem 1993) , thus skirting around **QS4** as a base logic. Our approach to the semantics of modal predicate logics is hence quite elementary, and it is hoped that it will provide a useful tool for philosophers, linguists or people working in AI.

In the next paragraph we start off by saying a few words about free logic and by defining the class of modal predicate logics we are going to investigate. In §3 we define the semantics and state a soundness theorem for our base logic **FK**. In §4 we prove a general completeness theorem by constructing a canonical model $\mathfrak{M}_{\mathcal{L}}$ for each modal predicate logic

$\mathcal{L}$. This proves completeness wrt so–called modal structures, which are frames (in the sense defined here) together with an interpretation of the relation symbols. In §5 we sketch how to obtain frame–completeness results by showing the canonicity of certain axioms. In the last paragraph we show how to obtain general frame–completeness by enriching the canonical frame by a suitable modal algebra of 'admissible interpretations'.

2 Free Logic and Free Modal Logic

Our language consists in some signature for predicate logic, with symbols for constants, functions and relations, and the set $Var := \{x_i : i \in \omega\}$ of variables. The set of formulae of this language is denoted by $\mathbb{L}$. $\mathbb{ML}$ is the language obtained by admitting any number of unary modal operators. For sake of exposition, we use just one modal operator, $\Box$. The base logic is not the usual quantified **K** (henceforth denoted by **QK**), but a weaker logic, denoted here by **FK**; its $\mathbb{L}$–fragment is denoted by $\mathbf{PFL}^{E!}$. For a more detailed treatment of the motivations behind the use of free logic the reader is referred to (Kutz 2000) and (Garson 1991). Here, a few brief remarks must suffice.

In the following, the existence symbol $E!$ abbreviates the formula $\exists y(y \doteq x)$, where y is a variable distinct from x. We first state the axioms of $\mathbf{PFL}^{E!}$. We then add the axioms for identity and then the axioms and rules of the propositional modal logic **K** to obtain **FK**.

Definition 2.1 (Axiomatization of $\boldsymbol{PFL}^{E!}$) Let ϕ and ψ be modal formulae. Then all formulae of the following type are axioms:

TAUTOLOGIES: All instances of classical propositional tautologies are axioms.

VACUOUS QUANTIFICATION: If x is not free in ϕ, then the formula $\phi \to \forall x\phi$ is an axiom.

UNIVERSAL DISTRIBUTIVITY: $\forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi)$.

E! 1: $\forall x E!(x)$.

E! 2: If y is free for x in $\phi(x, \bar{z})$ and $y \notin \bar{z}$, then $\forall x\phi(x, \bar{z}) \to (E!(y) \to \phi(y, \bar{z}))$ is an axiom.

Next, the following axioms for the identity symbol are added.

Definition 2.2 (Axioms for Equality) Let x and y be any variables and ϕ a formula such that x is free in ϕ and y is free for x in ϕ. $\phi(y//x)$ is shorthand for formulae which result from ϕ by replacing some (not necessarily all) occurrences of x by y. Then all formulae of the following type are axioms:

Self–Identity: $(x \doteq x)$

Leibniz' Law: $(x \doteq y) \to (\phi \to \phi(y//x))$

As usual, we have the rules of inference **Modus Ponens** (MP) and **Universal Generalization** ($\forall$). The axioms for $\mathbf{PFL}^{E!}$ together with the axioms for equality and the inference rules constitute the system $\mathbf{PFL}^{\doteq}$ of positive free logic with identity.

Before turning to the modal logic **FK**, a few short remarks on the use of free logic might be in order. First of all, since free logic is just a weakening of classical predicate logic, those who wish to deal with classical quantificational logic can simply install an extra axiom saying that all terms exist. This readily entails full classical universal instantiation. One can, on the other hand, use a *primitive* existence symbol $E!$, two (classical) quantifiers $\bigwedge$ and $\bigvee$ and define the (free) quantifiers $\forall$ and $\exists$ by restricting to the extension of the predicate $E!$ (the *domain of existence*).

Now, there is an important relationship between *existence* and *identity*, a fact which presumably goes back to Jaakko Hintikka.[1] If one formulates an axiom system with an existence symbol but without identity, the existence symbol is not *eliminable*, i.e. there is no formula $\phi(x)$, which does not contain the existence symbol, such that the formula $\phi(x) \leftrightarrow E!(x)$ is *provable*. This was shown by Bencivenga, Lambert and Meyer (cf. Bencivenga et al. 1982). But if the language is enriched by the identity symbol, the formula $E!(x)$ is provably equivalent to $\exists y(y \doteq x)$, whence $E!$ can be understood to be defined by this formula. In the literature, this fact has been called 'Hintikka's Theorem'.

Concerning the intrinsic reasons for using free logic, we just mention three basic points. Firstly, in standard quantified **K**, there are a number of theorems that do not seem to be valid under all possible interpretations. Consider for example the formula $\Box\exists x(x \doteq x)$, which states in the standard interpretation that it is *necessary that there are things*. Now it is perfectly imaginable that this claim be challenged by some philosopher. Also, the Converse Barcan Formula is easily proved in **QK**, while it is quite obviously disputable. Secondly, if one works for example in a varying domain setting, a formula like $\Diamond\phi(t)$ might contain a term t that denotes an object $\mathbf{a}$ in world w that does not exist in another accessible world v. If one then wants to assign a truth value to the formula $\Diamond\phi(t)$ at

[1] Compare his "Existential Presuppositions and Their Elimination" (Hintikka 1969).

world w, one has the choice between giving up the *principle of bivalence* or adopting free logic by assigning truth values also to formulae that contain terms denoting non–existing things. Thirdly, adopting classical predicate logic also blocks a proper treatment of non–denoting terms and definite descriptions which is particularly important in the modal setting.

We now turn to the definition of our modal base logic. The logic **FK** is obtained by adding the axiom–schema **Box–Distribution** and the rule **Necessitation** (MN). The only subtle point concerns the axioms for identity.

Definition 2.3 (Modal Axioms for Equality) Let x appear free in the formula ϕ and let y be a variable that does not appear free within the scope of a modal operator in ϕ and that is free for x in ϕ. The notation $\phi(y//x)$ is shorthand for a formula that results from ϕ by replacing some occurrences of x where x appears free but *not* within the scope of a modal operator by y. Within the scope of a modal operator, either *all* or *none* of the occurrences of the variable x have to be replaced by y. Then all formulae of the following type are axioms:

SELF–IDENTITY: $(x \doteq x)$

MODAL LEIBNIZ' LAW: $(x \doteq y) \to (\phi \to \phi(y//x))$

By means of the above definition of the identity axioms, we restrict the Quinean principle of the 'substitutability of identicals' insofar, as it is thus formulated so as to avoid all implications with respect to possible 'transworld identifications' of objects. That is to say, it avoids making assumptions about the behaviour of objects when one moves from one possible world to another. This suffices to establish 'classical circumstances' in the theory of a single world. Formulae on the other hand that entail 'transworld identifications' are assumed to be part of a given modal theory or otherwise of a logic stronger than the base logic.

By way of example, let us consider the following formulae, which are *not* correct instances of the Modal Leibniz' Law:

$$(x \doteq y) \to (\Box(x \doteq x) \to \Box(x \doteq y))$$

(Not all occurrences of the variable x are replaced by y, x is free within the scope of "$\Box$".)

$$(x \doteq y) \to (\Diamond(x \neq y) \to \Diamond(y \neq y))$$

(The variable y is free within the scope of a modal operator.)

$$(x \doteq y) \to (\exists y \Diamond(x \neq y) \to \exists y \Diamond(y \neq y))$$

(The variable y is not free for x.)

We illustrate this situation in the next section by giving countermodels for the formulae above.

However, the following formulae are correct instances:

$$(x \doteq y) \to (\Box(x \doteq x) \to \Box(y \doteq y))$$

(All occurrences of x are replaced by y.)

$$(x \doteq y) \to (\Diamond(x \not\doteq z) \to \Diamond(y \not\doteq z))$$

(The variable y is not free within the scope of a modal operator.)

$$(x \doteq y) \to ((x \doteq z) \to (y \doteq z))$$

(y is free for x and is not free within the scope of a modal operator.)

$$(x \doteq y) \to ((x \doteq x) \to (y \doteq x))$$

(y is free for x and is not free within the scope of a modal operator. Only one occurrence of x has been replaced by y.)

As the last examples have shown, we can easily establish that the equality symbol "$\doteq$" satisfies the axioms of an equivalence relation in any given world. The set of formulae derivable in the system **FK** is then denoted by $\mathbb{FK}$. The modal axioms for equality could also be thought of as a *minimal theory of identity* that accompanies every quantified modal logic. We now distinguish first and second order substitutions:

Definition 2.4 (First–Order Substitutions) Let $\phi(x, \bar{z})$ be a modal formula, in which x appears free and y is free for x. Then $\phi(y/x, \bar{z})$ is called a **first–order substitution instance**, if $\phi(y/x, \bar{z})$ is the result of replacing every free occurrence of x in ϕ by y.

Definition 2.5 (Second–order Substitutions) Let ϕ be a formula in which the n-place relation symbol P appears and let ψ be some modal formula. Then $(\psi/P)\phi$ is called a **second–order substitution instance**, if $(\psi/P)\phi$ is the result of replacing every occurrence of $P^n(\bar{y})$ in ϕ by $\psi(\bar{y}/\bar{x})$, possibly renaming some bound variables.

Note that assuming unrestricted second order substitution for a given logic $\mathcal{L}$ automatically extends the underlying modal theory of identity. For instance, given that $(x \doteq y) \to (P(x, x) \to P(x, y))$ is an admissible instance of Leibniz' Law, second order substitution yields $(x \doteq y) \to (\Box(x \doteq x) \to \Box(x \doteq y))$ and hence $(x \doteq y) \to \Box(x \doteq y)$. This is one of the motivations for distinguishing first- and second–order–closed logics.

Definition 2.6 (Modal Predicate Logics) A set of formulae $\mathcal{L}$ with $\mathbb{FK} \subseteq \mathcal{L} \subseteq \mathbb{ML}$ is called a **first–order closed modal predicate logic**, if it is closed under the rules (MN), ($\forall$) and (MP) and $\mathcal{L}$ is also closed under first–order substitutions. If $\mathcal{L}$ is additionally closed under second–order substitutions, it is called a **second–order closed modal predicate logic**. If we speak of a modal predicate logic $\mathcal{L}$ simpliciter, $\mathcal{L}$ is assumed to be at least first–order closed.

A natural solution to the above problem is therefore to deal with second–order closed logics without identity and to add a modal theory of identity, or, alternatively, to count the theory of identity as *part* of the logic and to restrict substitution in an appropriate way.

Note, that a second–order closed logic is a second–order logic in the sense, that predicate symbols are treated as second–order variables without having explicit second–order quantification. Hence, predicate variables are treated as being implicitly universally quantified.

3 Semantics

In the following, we very briefly introduce the relevant semantical concepts. First, we define first–order structures to interpret the free logical part of the system **FK**. We basically use the so–called inner–domain/outer–domain approach to free logic.

Definition 3.1 (First–Order Structures) A triple $\mathcal{S} = \langle \mathcal{U}_\mathcal{S}, \mathcal{D}_\mathcal{S}, \mathcal{I}_\mathcal{S} \rangle$ is called a **first–order structure** if $\mathcal{U}_\mathcal{S}$ is a *non–empty* set (the universe of the structure), $\mathcal{D}_\mathcal{S}$ a (possibly empty) subset of $\mathcal{U}_\mathcal{S}$ (the domain of existence) and $\mathcal{I}_\mathcal{S}$ an *interpretation*, which assigns to each n–place relation symbol a subset of the n–dimensional cartesian product of $\mathcal{U}_\mathcal{S}$. The class of all structures is then called $\mathcal{K}_{PFL}$.

The next step towards a definition of modal frames is to define an appropriate class of 'Counterpart–Relations'. The next definition is intended to replace the usual notion of the monotonicity of individual domains.

Definition 3.2 (Counterpart–Existence–Property) Let $\mathcal{S}$ and $\mathcal{T}$ be two structures and C a binary relation between $\mathcal{U}_\mathcal{S}$ and $\mathcal{U}_\mathcal{T}$. Then C has the **Counterpart–Existence–Property** wrt $\mathcal{S}$ (CE–Property for short), if for each element $\mathbf{a} \in \mathcal{U}_\mathcal{S}$ there is at least one element $\mathbf{b} \in \mathcal{U}_\mathcal{T}$, such that $\langle \mathbf{a}, \mathbf{b} \rangle \in C$. C is then said to be a *CE–relation*. $\mathcal{C}_{\mathcal{S},\mathcal{T}}$ stands for all CE–relations between $\mathcal{U}_\mathcal{S}$ and $\mathcal{U}_\mathcal{T}$.

This last definition does not only take care of the *bivalence* of the semantics, it is also necessary to establish *normality*, as a look at the soundness prove below will reveal.

Definition 3.3 (Modal Structures and Frames) A pair $\mathfrak{f} = \langle \mathcal{W}, \mathcal{C} \rangle$ is a **modal structure**, if $\mathcal{W} \neq \emptyset$ is a *non–empty* set of first–order structures and $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{W}}$ is a subset of the set $\mathcal{C}_{\mathcal{W}}$ of all CE–relations between (universes of) structures from $\mathcal{W}$. Members $\mathcal{S}$ of $\mathcal{W}$ are also called **(possible) worlds**. $\mathfrak{Co}_{\mathcal{S},\mathcal{T}}$ denotes the set of all CE–relations between $\mathcal{S}$ and $\mathcal{T}$ in the modal structure $\mathfrak{f}$. The class of all modal structures is called $\mathfrak{fr}\mathfrak{K}$. If the interpretation–function of the modal structure $\mathfrak{f}$ is omitted, we denote the result by $\mathfrak{F} = \langle \mathcal{U}, \mathcal{C} \rangle$ (where $\mathcal{U}$ is now just a family of pairs of the form $\langle U_w, D_w \rangle$ with $w \in \mathcal{W}$) and call $\mathfrak{F}$ a **modal frame**. The class of all modal frames will be denoted by $\mathfrak{Fr}\mathfrak{K}$.

We say that $\mathcal{S}$ **sees** $\mathcal{T}$ in $\mathfrak{f}$ if there is a CE–relation from $\mathcal{S}$ to $\mathcal{T}$ in $\mathfrak{f}$. A valuation is a function υ which assigns to each variable x and possible world $\mathcal{S}$ an element from the universe $\mathcal{U}_{\mathcal{S}}$ of $\mathcal{S}$. We write $\upsilon_{\mathcal{S}}$ for the valuation υ at $\mathcal{S}$. A **modal model** $\mathfrak{M} = \langle \mathfrak{f}, \upsilon \rangle$ is a modal structure $\mathfrak{f}$ together with a valuation υ. Finally, an **existential x–variant** of a valuation υ in the world $\mathcal{S}$ is a valuation function $\widetilde{\upsilon}$ which is like υ except that $\widetilde{\upsilon}$ assigns some element from $\mathcal{D}_{\mathcal{S}}$ to the variable x at $\mathcal{S}$.

Definition 3.4 (Truth in a Modal Model) Let $\phi(y_1, \ldots, y_n)$ and $\psi(z_1, \ldots, z_m)$ be modal formulae with the free variables $y_1, \ldots, y_n$ and $z_1, \ldots, z_m$, respectively. Let $\mathfrak{f}$ be a modal structure, $\mathcal{S}$ a possible world and let υ be a valuation. We define:

(a) $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash x_i \doteq x_j :\Longleftrightarrow \upsilon_{\mathcal{S}}(x_i) = \upsilon_{\mathcal{S}}(x_j)$ in $\mathcal{U}_{\mathcal{S}}$.

(b) $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash R(y_1, \ldots, y_n) :\Longleftrightarrow \langle \upsilon_{\mathcal{S}}(y_1), \ldots, \upsilon_{\mathcal{S}}(y_n) \rangle \in \mathcal{I}_{\mathcal{S}}(R)$.

(c) $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \neg\phi :\Longleftrightarrow \langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \nvDash \phi$.

(d) $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \phi \wedge \psi :\Longleftrightarrow \langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \phi$ and $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \psi$.

(e) $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \Diamond\phi(y_1, \ldots, y_n) :\Longleftrightarrow$ there is $\mathcal{S} \xrightarrow{C} \mathcal{T}$ and a $\langle y_1, \ldots, y_n \rangle$-variant $\widetilde{\upsilon}$, such that $\langle \upsilon_{\mathcal{S}}(y_i), \widetilde{\upsilon}_{\mathcal{T}}(y_i) \rangle \in C$ for $i = 1, \ldots, n$ and $\langle \mathfrak{f}, \widetilde{\upsilon}, \mathcal{T} \rangle \vDash \phi(y_1, \ldots, y_n)$.

(f) $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \exists x \phi(x) :\Longleftrightarrow$ there is an existential x–variant $\widetilde{\upsilon}$, such that $\langle \mathfrak{f}, \widetilde{\upsilon}, \mathcal{S} \rangle \vDash \phi(x)$.

Now the usual soundness theorems like e.g. a Coincidence Lemma can be proved, of which we shall make free use below. In particular, we can prove the following Soundness Theorem for **FK**. The only subtle points concern the normality of **FK** and the modal equality axioms.

Theorem 3.5 (Soundness) *Let $\mathfrak{F}$ be an arbitrary modal frame. Then it holds that $\mathfrak{F} \vDash$* **FK**.

Proof. We have to prove that every formula that is derivable in **FK** is true in any modal model $\mathfrak{M}$. We only consider Leibniz' Law and Box–Distribution. The validity of the other axiom–schemes is easily verified.

MODAL LEIBNIZ' LAW Inductively we show the following: *Suppose that ϕ is an arbitrary modal formula and $\widetilde{\phi} := (x \doteq y) \to (\phi \to \phi(y//x))$ is an **admissible** instance of Leibniz' Law. Then $\widetilde{\phi}$ is valid.* From this, Leibniz' Law follows.

(i) This claim is trivial for atomic formulae and easy to verify for formulae of the form $\neg\psi$ or $(\psi \wedge \chi)$.

(ii) Let $\phi(x, y, \bar{z}) = \exists w \psi(w, x, y, \bar{z})$ and $\widetilde{\phi}(x, y, \bar{z})$ be an admissible instance of Leibniz' Law. Then

$$(x \doteq y) \to (\psi(w, x, y, \bar{z}) \to \psi(w, y//x, y, \bar{z})) \qquad (\star)$$

is also admissible, hence valid by induction hypotheses. Now suppose that $\langle v, \mathcal{S}\rangle \vDash (x \doteq y) \wedge \exists w \psi(w, x, y, \bar{z})$ in some model $\mathfrak{M}$. Then $v_{\mathcal{S}}(x) = v_{\mathcal{S}}(y)$ and $\langle \widetilde{v}, \mathcal{S}\rangle \vDash \psi(w, x, y, \bar{z})$ for some w–variant $\widetilde{v}$ of v. Because the variable w is different from x and y, $\widetilde{v}_{\mathcal{S}}(x) = \widetilde{v}_{\mathcal{S}}(y)$, whence $(\star)$ implies $\langle \widetilde{v}, \mathcal{S}\rangle \vDash \psi(w, y//x, y, \bar{z})$. It follows that $\langle v, \mathcal{S}\rangle \vDash \exists w \psi(w, y//x, y, \bar{z})$.

(iii) Let $\phi(x, \bar{z}) = \Diamond\psi(x, \bar{z})$ and $\widetilde{\phi}(x, \bar{z})$ be admissible. Then, by definition of 'admissible instance', $y \notin \bar{z}$ and x is within the scope of a modal operator. Therefore $\widetilde{\phi}(x, \bar{z})$ is of the form $(x \doteq y) \to (\Diamond\psi(x, \bar{z}) \to \Diamond\psi(y, \bar{z}))$, where every occurrence of x in $\psi(y, \bar{z}))$ is replaced by y. Now let $\mathfrak{M}$ be a model and $\mathcal{S}$ a world such that $\langle \mathcal{S}, v\rangle \vDash (x \doteq y) \wedge \Diamond\psi(x, \bar{z})$. Then $v_{\mathcal{S}}(x) = v_{\mathcal{S}}(y)$ and there is a world $\mathcal{T}$ and a $(x, \bar{z})$–variant $\widetilde{v}$, such that $\mathcal{S} \xrightarrow{C} \mathcal{T}$, $\langle v_{\mathcal{S}}(x), \widetilde{v}_{\mathcal{T}}(x)\rangle \in C$ and $\langle v_{\mathcal{S}}(z_i), \widetilde{v}_{\mathcal{T}}(z_i)\rangle \in C$ for all $z_i \in \bar{z}$ and $\langle \widetilde{v}, \mathcal{T}\rangle \vDash \psi(x, \bar{z})$. Now let $\widehat{v}$ be a $(y, \bar{z})$–variant of v such that $\widehat{v}_{\mathcal{T}}(y) = \widetilde{v}_{\mathcal{T}}(x)$ and $\widehat{v}_{\mathcal{T}}(z_i) = \widetilde{v}_{\mathcal{T}}(z_i)$. It may be the case that $\widetilde{v}_{\mathcal{T}}(x) \neq \widetilde{v}_{\mathcal{T}}(z_i)$ for all i. Hence the above definition of $\widehat{v}$ is only admissible, since—by assumption—y is different from all z_i. So the valuations $\widehat{v}$ and $\widetilde{v}$ coincide at $\mathcal{T}$ on the free variables of ψ and y is free for x in $\psi(x, \bar{z})$. By the Coincidence Lemma it follows that $\langle \widehat{v}, \mathcal{T}\rangle \vDash \psi(y, \bar{z})$ and because $v_{\mathcal{S}}(x) = v_{\mathcal{S}}(y)$ and $\widehat{v}$ is a $(y, \bar{z})$–variant of $v_{\mathcal{T}}$ with $\langle v_{\mathcal{S}}(u), \widehat{v}_{\mathcal{T}}(u) \in C$ for all $u \in y \cup \bar{z}$, we have that $\langle v, \mathcal{S}\rangle \vDash \Diamond\psi(y, \bar{z})$.

BOX–DISTRIBUTION Let $\phi(\bar{x})$ and $\psi(\bar{y})$ be modal formulae with free variables $\bar{x}$ and $\bar{y}$ respectively and suppose that $\langle v, \mathcal{S}\rangle \vDash \Box(\phi(\bar{x}) \to \psi(\bar{y})) \wedge \Box\phi(\bar{x})$. Suppose wlog that there is a world $\mathcal{T}$ in a modal model $\mathfrak{M}$ and a CE–relation C, such that $\mathcal{S} \xrightarrow{C} \mathcal{T}$. Now let $\widetilde{v}$ be any $\bar{y}$–variant of v such that $\langle v_{\mathcal{S}}(y_i), \widetilde{v}_{\mathcal{T}}(y_i)\rangle \in C$ for all $y_i \in \bar{y}$. Because C is a CE–relation, there is—for every variable $x_j \in \bar{x} - \bar{y}$—an element $\mathbf{a}_j \in \mathcal{U}_{\mathcal{T}}$, such that $\langle \upsilon_{\mathcal{S}}(x_j), \mathbf{a}_j \rangle \in C$. Now define a $\bar{x} \cup \bar{y}$–variant $\widehat{\upsilon}$ of υ by:

$$\widehat{\upsilon}_{\mathcal{T}}(x) = \begin{cases} \mathbf{a}_i & : \text{ if } x = x_i \in \bar{x} - \bar{y} \\ \widetilde{\upsilon}_{\mathcal{T}}(x) & : \text{ otherwise} \end{cases}$$

By assumption $\langle \widehat{\upsilon}, \mathcal{T} \rangle \vDash (\phi(\bar{x}) \to \psi(\bar{y})) \wedge \phi(\bar{x})$, hence $\langle \widehat{\upsilon}, \mathcal{T} \rangle \vDash \psi(\bar{y})$. Now the valuations $\widehat{\upsilon}$ and $\widetilde{\upsilon}$ coincide at $\mathcal{T}$ on the variables in $\bar{y}$, whence by the Coincidence Lemma $\langle \widetilde{\upsilon}, \mathcal{T} \rangle \vDash \psi(\bar{y})$.

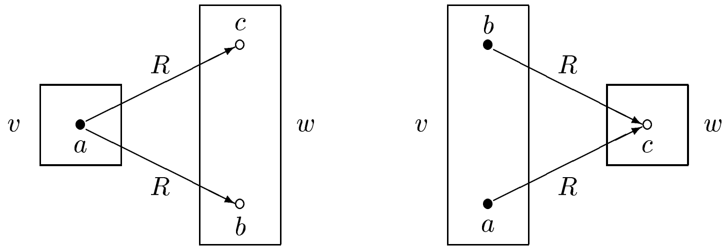
Finally, it is straightforwardly checked that the rules of inference preserve the validity of the axioms of **FK**. $\dashv$

We now come to the promised examples. In the following pictures, a world is represented by a square box with individuals being represented by a bullet if they exist at that world and by a circle otherwise.

FIGURE 1 Refuting the Barcan Formula and its Converse

In figure 1, the objects a and b are related via a CE-Relation R. The model on the left refutes the Barcan Formula $\Diamond\exists x(x \doteq x) \to \exists x\Diamond(x \doteq x)$ and also the Necessity of Fictionality $\neg E!(x) \to \Box\neg E!(x)$. Correspondingly, the model on the right refutes the Converse Barcan Formula and the Necessity of Existence.

FIGURE 2 Fusing and Splicing of Objects



In figure 2, the left model refutes the Necessity of Identity $(x \doteq y) \to \Box(x \doteq y)$, whereas the model on the right refutes the Necessity of

Distinctness $(x \neq y) \to \Box(x \neq y)$. The former also relates to "Pierre's Puzzle" (Kripke 1979). Think of world v as the actual world and of world w as Pierre's "World of Beliefs". Then Pierre's seemingly inconsistent beliefs about the actual city denoted in the real world by constants like `London` and `Londres` can be consistently modelled as a de re belief about two *distinct* (fictional) objects in his Belief–World.

4 Canonical Models

Definition 4.1 (Free Henkin–Types) A maximally consistent set of modal formulae Δ is called a **Free Henkin–type** if it also satisfies the following condition:

(H) For every $\exists$–formula $\exists x\phi(x, \bar{z}) \in \Delta$ there is a variable y such that $\phi(y, \bar{z}) \in \Delta$, $E!(y) \in \Delta$ and y is free for x in $\phi(x, \bar{z})$.

The set of all free Henkin–types wrt the logic $\mathcal{L}$ will be denoted by $FHen_{\mathcal{L}}$.

In the sequel, we often use substitutions $f : Var \longrightarrow Var$ that are totally defined and injective. We will call them **faithful substitutions** and denote the set of all these functions by $\mathbb{F}$.

Lemma 4.2 (Existence of Free Henkin–Types) *For every consistent set Δ and any two* finite *sequences $y_1, \ldots, y_m$ and $z_1, \ldots, z_m$ of m pairwise distinct variables there is a faithful substitution $f \in \mathbb{F}$ such that $f(y_i) = z_i$ for $i = 1, \ldots, m$, and a free Henkin–type Γ such that*

$$\Delta^f := \{\phi^f(\bar{v}) \ : \ \phi(\bar{u}) \in \Delta \text{ and } f(\bar{u}) = \bar{v}\} \subseteq \Gamma,$$

where $f(\bar{u}) = \bar{v}$ abbreviates $f(u_i) = v_i, i = 1, \ldots, n$.

Proof. Fix an enumeration $\pi : \omega \to \mathbb{ML}$. We define a sequence $\langle \Gamma_{-1}, \langle \Gamma_i : i \in \omega \rangle \rangle$ in the following way. Because the given lists of variables are finite, there are $k, l \in \omega$, such that $\{y_1, \ldots, y_m\} \subset \{x_1, \ldots, x_k\}$ and $\{z_1, \ldots, z_m\} \subset \{x_1, \ldots, x_l\}$. Now let f be defined as follows:

$$f(x) = \begin{cases} z_i & : \text{ if } x = y_i \text{ (for } i \in \{1, \ldots, m\}) \\ x_{l+2i+1} & : \text{ if } x = x_i \text{ and } x \neq y_i \text{ for } i = 1, \ldots, m \end{cases}$$

Since the y_i are all distinct, this (total) function is faithful. In particular, $range(f) \subset \{z_1, \ldots, z_n\} \cup \{x_{l+2i+1} \ : \ i \in \omega\}$, hence infinitely many variables (namely those of the form $x_{l+2\cdot(i+1)}$ for $i \in \omega$) are *not* within the range of f. Put $\Gamma_{-1} := \Delta^f$. Define Γ_n inductively as follows:

(i) If $\Gamma_{n-1} \cup \{\pi(n)\}$ is $\mathcal{L}$–consistent and $\pi(n)$ is not an $\exists$–formula, put $\Gamma_n := \Gamma_{n-1} \cup \{\pi(n)\}$.

(ii) If $\Gamma_{n-1} \cup \{\pi(n)\}$ is $\mathcal{L}$–consistent and $\pi(n) = \exists x \phi(x, x_{i_1}, \dots, x_{i_j})$ is an $\exists$–formula, put $\Gamma_n := \{\phi(x_i, x_{i_1}, \dots, x_{i_j}), E!(x_i), \pi(n)\} \cup \Gamma_{n-1}$, where $x_i \in \mathit{Var}$ is the first variable in the standard–enumeration, which is not under the (free or bound) variables occurring in $\Gamma_{n-1} \cup \pi(n)$. (Alternatively, choose x_i as $x_{l+2(n+1)}$.)

(iii) Finally, if $\Gamma_{n-1} \cup \{\pi(n)\}$ is $\mathcal{L}$–inconsistent, we define $\Gamma_n := \Gamma_{n-1} \cup \{\neg\pi(n)\}$. (Notice that $\neg\pi(n)$ is then not an $\exists$–formula, even if it might be $\mathcal{L}$–equivalent to one.)

Γ_n is easily seen to be consistent. Now set $\Gamma := \bigcup_{i \in \{-1\} \cup \omega} \Gamma_i$. This is a complete type. Moreover, by construction Γ is a free Henkin–Type and in particular $\Delta^f \subset \Gamma$. $\dashv$

Definition 4.3 (Canonical Universe of Discourse) Let Δ be a free Henkin–type. Define an equivalence relation $\sim_\Delta$ on variables by putting $x \sim_\Delta y$ iff $(x \doteq y) \in \Delta$. The universe of the structure $\mathcal{S}_\Delta$ is then defined as: $\mathcal{U}_\Delta := \{[x_i]_\Delta^\sim \ : \ x_i \in \mathit{Var}\}$ and the domain of quantification as $\mathcal{D}_\Delta := \{[x_i]_\Delta^\sim \ : \ x_i \in \mathit{Var} \text{ and } E!(x_i) \in \Delta\}$.

Definition 4.4 (Canonical Interpretation) Let Δ be a free Henkin–type, R an n–place relation symbol and $y_1, \dots, y_n$ n variables. We define an n–place relation R^Δ on the n–dimensional cartesian product of $\mathcal{U}_\Delta$ by setting:

$$\langle [y_1]_\Delta, \dots, [y_n]_\Delta \rangle \in R^\Delta :\Longleftrightarrow R(y_1, \dots, y_n) \in \Delta.$$

The **canonical interpretation** is then $\mathcal{I}_\Delta(R) := R^\Delta \subset \mathcal{U}^n$.

It is easily shown that this is well–defined. Next we define the notion of a canonical structure.

Definition 4.5 (Canonical Structures) If Δ is a Free Henkin–Type, we define the **canonical structure** $\mathcal{S}_\Delta$ as the triple $\langle \mathcal{U}_\Delta, \mathcal{D}_\Delta, \mathcal{I}_\Delta \rangle$. The class $\{\mathcal{S}_\Delta \mid \Delta \in \mathit{FHen}_\mathcal{L}\}$ of all canonical structures is then denoted by $\mathcal{W}_\mathcal{L}$.

Definition 4.6 (Canonical Valuation) Let Δ be a free Henkin–type and let $\mathcal{S}_\Delta$ be the associated canonical structure. Then the **canonical valuation** $v_\Delta : \mathit{Var} \longrightarrow \mathcal{U}_\Delta$ is defined by $v_\Delta(x_i) := [x_i]_\Delta$.

Definition 4.7 (Canonical Counterpart–Relations) Let f be any faithful substitution and Δ and Γ be two free Henkin–types. We then define the expression $C_f := \{\langle [x]_\Delta, [y]_\Gamma \rangle \ : \ f(x) = y\}$ and finally put:

$$(\Delta^\Box)^f := \{\phi^f(\bar{z}) \ : \ \Box\phi(\bar{y}) \in \Delta \text{ and } f(y_i) = z_i, i = 1, \dots, n\}.$$

Finally define: $\mathcal{S}_\Delta \xrightarrow{C_f} \mathcal{S}_\Gamma :\Longleftrightarrow (\Delta^\Box)^f \subset \Gamma$ (or briefly $\Delta \xrightarrow{C_f} \Gamma$). The class $\mathcal{C}_\mathcal{L}$ of all **canonical Counterpart–relations** of the canonical modal structure is then given by:

$$\mathcal{C}_\mathcal{L} := \{C_f \ : \ f \in \mathbb{F},\ \Delta \xrightarrow{C_f} \Gamma \textit{ and } \Delta, \Gamma \in \textit{FHen}_\mathcal{L}\}.$$

By constructing appropriate free Henkin–types, it is easily seen that canonical Counterpart–relations are in general neither functional nor injective or surjective etc. We now define the notion of a canonical modal structure and give the central result in Theorem 4.9.

Definition 4.8 (Canonical Modal Structure/Model) The **canonical modal structure** $\mathfrak{f}_\mathcal{L}$ is given by the pair $\langle \mathcal{W}_\mathcal{L}, \mathcal{C}_\mathcal{L}\rangle$. The **canonical modal model** is then $\mathfrak{M}_\mathcal{L} = \langle \mathfrak{f}_\mathcal{L}, \upsilon_\mathcal{L}\rangle$, where $\upsilon_\mathcal{L}$ denotes the canonical valuation.

Theorem 4.9 (Fundamental Theorem) *Let $\mathcal{L}$ be any modal predicate logic, ϕ be an arbitrary modal formula, $\mathfrak{f}_\mathcal{L}$ the canonical modal structure, $\mathcal{S}_\Delta$ any world and $\upsilon_\mathcal{L}$ the canonical valuation. Then*

$$\langle \mathfrak{f}_\mathcal{L}, \upsilon_\mathcal{L}, \mathcal{S}_\Delta\rangle \vDash \phi \Longleftrightarrow \phi \in \Delta.$$

Proof. The proof is by induction on the construction of ϕ. We perform only the problematic steps.

(i) Let $\phi = \exists x \psi(x, \bar{y})$. $\langle \mathfrak{f}_\mathcal{L}, \upsilon_\mathcal{L}, \mathcal{S}_\Delta\rangle \vDash \exists x\psi(x,\bar{y}) :\Longleftrightarrow$ *there exists an existential x–variant $\widetilde{\upsilon}$ such that* $\langle \widetilde{\upsilon}, \mathcal{S}_\Delta\rangle \vDash \psi(x,\bar{y})$. By definition this is the case if and only if there is a valuation $\widetilde{\upsilon}$ with $\widetilde{\upsilon}_\Delta(x) = [z]_\Delta \in \mathcal{D}_\Delta$ such that $\langle \upsilon_x^{[z]_\Delta}, \mathcal{S}_\Delta\rangle \vDash \psi(x,\bar{y})$. Here the valuations υ and $\upsilon_x^{[z]_\Delta}$ coincide at $\mathcal{S}_\Delta$ on all variables save possibly x and $\upsilon_\Delta(z) = [z]_\Delta = \upsilon_x^{[z]_\Delta}(x)$. Now let $\widetilde{\psi}(x,\bar{y})$ be a bound variant of $\psi(x,\bar{y})$, such that z is free for x in $\widetilde{\psi}(x,\bar{y})$. By the Coincidence Lemma it follows that $\langle \upsilon, \mathcal{S}_\Delta\rangle \vDash \widetilde{\psi}(z,\bar{y})$ *and* $[z]_\Delta \in \mathcal{D}_\Delta$. But by assumption and definition this means that there is a variable z such that $E!(z) \in \Delta$ and $\widetilde{\psi}(z,\bar{y}) \in \Delta$. Because $E!(z) \wedge \widetilde{\psi}(z,\bar{y}) \to \exists z\widetilde{\psi}(z,\bar{y}) \in \mathcal{L}$ and Δ is a complete type it follows that $\exists z\widetilde{\psi}(z,\bar{y}) \in \Delta$. But then, since complete types are closed under renaming of bound variables, $\exists x\psi(x,\bar{y})$ is in Δ as well. Conversely, assume that $\exists x\psi(x,\bar{y}) \in \Delta$. Then, since Δ is a free Henkin–type, there is a variable z such that $E!(z) \in \Delta$ and $\psi(z,\bar{y}) \in \Delta$. Now we can reverse the last argument. This proves part (i).

(ii) Assume that $\phi(x_1, \ldots, x_n) = \Diamond\psi(x_1,\ldots,x_n)$ and that the claim holds for ψ.

Assume first, that $\langle \upsilon, \mathcal{S}_\Delta \rangle \vDash \Diamond\psi(x_1, \ldots, x_n)$. Then, by definition, there is a free Henkin–type Γ and a $C_f \in \mathfrak{Co}_{\Delta,\Gamma}$, such that $(\Delta^\Box)^f \subset \Gamma$. Furthermore there are variables $y_1, \ldots, y_n$, such that $\langle [x_i]_\Delta, [y_i]_\Gamma \rangle \in C_f$ and $\langle \mathcal{S}_\Gamma, \upsilon^{[y_i]_\Gamma}_{x_i} \rangle \vDash \psi(x_1, \ldots, x_n)$. But then there are variables $u_i \in [x_i]_\Delta$ and $v_i \in [y_i]_\Gamma$ such that $f(u_i) = v_i$ $(i = 1, \ldots, n)$. Then, in particular, $[u_i]_\Delta = [x_i]_\Delta$ and $[v_i]_\Gamma = [y_i]_\Gamma$ and hence $\langle [u_i], [v_i] \rangle \in C_f$ for $i = 1, \ldots, n$. It follows that $\langle \upsilon^{[v_i]_\Gamma}_{x_i}, \mathcal{S}_\Gamma \rangle \vDash \psi(x_1, \ldots, x_n)$. The variables v_i might not be free for x_i in $\psi(x_1, \ldots, x_n)$. Therefore let g be a faithful substitution such that $g(x_i) = v_i$ (and g arbitrary otherwise). Then $\langle \upsilon^{[v_i]_\Gamma}_{x_i} \circ g^{-1}, \mathcal{S}_\Gamma) \rangle \vDash \psi^g(v_1, \ldots, v_n)$, i.e. $\langle \upsilon_\mathcal{L}, \mathcal{S}_\Gamma \rangle \vDash \psi^g(v_1, \ldots, v_n)$. By assumption it follows that

$$\psi^g(v_1, \ldots, v_n) \in \Gamma \qquad (*).$$

Now assume that $\Diamond\psi(x_1, \ldots, x_n) \notin \Delta$. Because $[x_i]_\Delta = [u_i]_\Delta$ for $i = 1, \ldots, n$, it follows that $(x_i \doteq u_i) \in \Delta$ for $i = 1, \ldots, n$. Moreover, let h be faithful with $h(x_i) = u_i$ for $i = 1, \ldots, n$. It follows that $\Diamond\psi(x_1, \ldots, x_n) \in \Delta \iff \Diamond\psi^h(u_1, \ldots, u_n) \in \Delta$, whence $\Diamond\psi^h(u_1, \ldots, u_n) \notin \Delta$. Then $\Box\neg\psi^h(u_1, \ldots, u_n) \in \Delta$ and since $(\Delta^\Box)^f \subset \Gamma$ we also deduce that we have $(\neg\psi^h)^f(f(u_1), \ldots, f(u_n)) \in \Gamma$, i.e. that

$$\neg\psi^{f \circ h}(v_1, \ldots, v_n) \in \Gamma \qquad (**).$$

But since the formulae $(*)$ and $(**)$ differ only in the choice of bound variables, this immediately yields a contradiction.

Conversely, assume that $\Diamond\psi(x_1, \ldots, x_n) \in \Delta$. Then there is a possible world Γ and a canonical Counterpart–relation C_f in $\mathfrak{Co}_{\Delta,\Gamma}$, such that $\psi^f(y_1, \ldots, y_n) \in \Gamma$. Then $(\Delta^\Box)^f \subset \Gamma$, where $f(x_i) = y_i$ for $i = 1, \ldots, n$. Hence $\Delta \xrightarrow{C_f} \Gamma$ and $\langle [x_i]_\Delta, [y_i]_\Gamma \rangle \in C_f$ for $i = 1, \ldots, n$. By assumption it follows that $\langle \upsilon, \mathcal{S}_\Gamma \rangle \vDash \psi^f(y_1, \ldots, y_n)$ and hence $\langle \upsilon \circ f, \mathcal{S}_\Gamma \rangle \vDash \psi(x_1, \ldots, x_n)$. (Here $(\upsilon \circ f)_\Theta = \upsilon$ for all $\Theta \neq \Gamma$, and $(\upsilon \circ f)_\Gamma := \upsilon_\Gamma \circ f$.) But then there is a $\langle x_1, .., x_n \rangle$–variant $\widetilde{\upsilon}$ of υ, namely $\widetilde{\upsilon}_\Gamma(x_i) = [f(x_i)]_\Gamma = [y_i]_\Gamma$, such that by the Coincidence Lemma $\langle \widetilde{\upsilon}, \mathcal{S}_\Gamma \rangle \vDash \psi(x_1, \ldots, x_n)$ and $\langle \upsilon_\Delta(x_i), \widetilde{\upsilon}_\Gamma(x_i) \rangle \in C_f$. Hence $\langle \upsilon, \mathcal{S}_\Delta \rangle \vDash \Diamond\psi(x_1, \ldots, x_n)$, which completes the proof of the theorem. $\dashv$

As usual, $\mathfrak{M} \vDash \phi$ (where $\mathfrak{M} = \langle \mathfrak{f}, \upsilon \rangle$) if for all worlds $\mathcal{S}$, $\langle \mathfrak{f}, \upsilon, \mathcal{S} \rangle \vDash \phi$. Put $Th(\mathfrak{M}) := \{\phi \in \mathbb{ML} : \mathfrak{M} \vDash \phi\}$. Furthermore, $\mathfrak{f} \vDash \phi$ if for all models $\mathfrak{M}$ based on $\mathfrak{f}$ (i.e. for all valuations in $\mathfrak{f}$), $\mathfrak{M} \vDash \phi$ holds. Finally, $\mathfrak{F} \vDash \phi$ if for all interpretations $\mathcal{I}$ in $\mathfrak{F}$, $\mathfrak{f} \vDash \phi$, where $\mathfrak{f} = \langle \mathfrak{F}, \mathcal{I} \rangle$. On the basis of this it is not hard to show the following.

Theorem 4.10 (Canonical Model Theorem) *Let $\mathcal{L}$ be a modal predicate logic. Then $Th(\mathfrak{M}_\mathcal{L}) = \mathcal{L}$. In particular, the canonical valuation* *is dispensable, i.e.* $\mathcal{L} = Th(\mathfrak{f}_{\mathcal{L}})$.

However, this only establishes completeness wrt models and not frames.

5 Canonicity and Frame–Completeness

Definition 5.1 (Frame–Completeness) We say that a modal predicate logic $\mathcal{L}$ is **frame–complete**, if there is a class $\mathbb{K}$ of modal frames such that $\mathcal{L} = \bigcap_{\mathfrak{F}\in\mathbb{K}} Th(\mathfrak{F})$.

Definition 5.2 (Canonicity) A logic $\mathcal{L}$ is said to be **canonical**, if it is valid on the canonical frame $\mathfrak{F}_{\mathcal{L}}$, i.e. if $\mathfrak{F}_{\mathcal{L}} \vDash \mathcal{L}$.

The following are now immediate.

Theorem 5.3 *Every canonical logic is frame–complete.*

Proof. Immediately from Theorem 4.10. $\dashv$

Corollary 5.4 *The logic* **FK** *is frame–complete with respect to the class* $\mathfrak{Fr}\mathfrak{K}$ *of all frames.*

Proof. By 3.5 and 4.10. $\dashv$

It is possible to establish a correspondence between properties of Kripke–structures and certain axioms in the usual way but also between properties of the $\mathcal{C}$–relations and axioms. Examples are the functionality or injectivity of all CE–relations of a model which correspond respectively to the validity of the axiom–schemes $(x \doteq y) \to \Box(x \doteq y)$ (Necessity of Identity) and $(x \neq y) \to \Box(x \neq y)$ (Necessity of Distinctness). If the former axiom–schema is denoted by (N) then specializing on **QS4N** (where 'Q' signalizes that we also specialized to classical predicate logic), we obtain the $\mathcal{C}$–sets of (Ghilardi 1991). We further illustrate this situation and the kind of correspondence that obtains by the following examples.

Definition 5.5 (Prominent Schemes) Let x and y be variables and let ϕ be a modal formula:

(NoI)	$(x \doteq y) \to \Box(x \doteq y)$	(Necessity of Identity)
(NoD)	$(x \neq y) \to \Box(x \neq y)$	(Necessity of Distinctness)
(NoE)	$E!(x) \to \Box E!(x)$	(Necessity of Existence)
(NoF)	$\neg E!(x) \to \Box\neg E!(x)$	(Necessity of Fictionality)
(BF)	$\forall x\Box\phi \to \Box\forall x\phi$	(Barcan–formulae)

(CBF)	$\Box\forall x\phi \to \forall x\Box\phi$	(Converse Barcan–formulae)
(T)	$\Box\phi \to \phi$	(T–Schema)
(4)	$\Box\phi \to \Box\Box\phi$	(4–Schema)

In the following we define various properties of frames and state some soundness and canonicity–results.

Definition 5.6 (Properties of Frames)
(a) A frame $\mathfrak{F}$ is called **functional** or **fission–free**, if all $C \in \mathcal{C}_{\mathfrak{F}}$ are functional.
(b) $\mathfrak{F}$ is called **fusion–free** or also **injektiv**, if all $C \in \mathcal{C}_{\mathfrak{F}}$ are injective.
(c) $\mathfrak{F}$ is called **existentially faithful**, if for all $\mathcal{S}$ and $\mathcal{T}$ and all $C \in \mathfrak{Co}_{\mathcal{S},\mathcal{T}}$: If $\mathbf{a} \in \mathcal{D}_{\mathcal{S}}$ and $\langle \mathbf{a}, \mathbf{b} \rangle \in C$ then $\mathbf{b} \in \mathcal{D}_{\mathcal{T}}$.
(d) We call $\mathfrak{F}$ **existentially friendly**, if for all worlds $\mathcal{S}$ and $\mathcal{T}$ such that $\mathfrak{Co}_{\mathcal{S},\mathcal{T}} \neq \emptyset$: If $\mathbf{b} \in \mathcal{D}_{\mathcal{T}}$, then there is a $\mathbf{a} \in \mathcal{D}_{\mathcal{S}}$ and a $C \in \mathfrak{Co}_{\mathcal{S},\mathcal{T}}$, such that $\langle \mathbf{a}, \mathbf{b} \rangle \in C$.
(e) If for all possible worlds $\mathcal{S}$ and $\mathcal{T}$ it holds that $\mathcal{C}(\mathcal{U}_{\mathcal{S}} \backslash \mathcal{D}_{\mathcal{S}}) \subseteq \mathcal{U}_{\mathcal{T}} \backslash \mathcal{D}_{\mathcal{T}}$ (meaning that for all $C \in \mathcal{C}_{\mathcal{S},\mathcal{T}} : \langle \mathbf{a}, \mathbf{b} \rangle \in C$ and $\mathbf{a} \in \mathcal{U}_{\mathcal{S}} \setminus \mathcal{D}_{\mathcal{S}}$ implies that $\mathbf{b} \in \mathcal{U}_{\mathcal{T}} \setminus \mathcal{D}_{\mathcal{T}}$), then $\mathfrak{F}$ is called **fictionally faithful**.
(f) A frame $\mathfrak{F}$ is said to be **locally reflexive**, if for every world $\mathcal{S}$, every natural number n and every n–tupel $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ of elements from $\mathcal{U}_{\mathcal{S}}$ there is a relation $C \in \mathfrak{Co}_{\mathcal{S},\mathcal{S}}$ such that $C \supset \{\langle \mathbf{a}_i, \mathbf{a}_i \rangle \mid i = 1, \ldots, n\}$.
(g) $\mathfrak{F}$ is then called **reflexive**, if it is locally–reflexive and for every world $\mathcal{S}$ and every n–tuple in $\mathcal{U}_{\mathcal{S}}$ the same $C \in \mathfrak{Co}_{\mathcal{S},\mathcal{S}}$ can be chosen, i.e. $C \supset \{\langle \mathbf{a}, \mathbf{a} \rangle \mid \mathbf{a} \in \mathcal{U}_{\mathcal{S}}\}$.
(h) $\mathfrak{F}$ is called **locally transitive**, if for every pair $\mathcal{S} \xrightarrow{C} \mathcal{T}$ and $\mathcal{T} \xrightarrow{\widehat{C}} \mathcal{R}$, every $n \in \mathbb{N}$ and every triple $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$, $\langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ and $\langle \mathbf{c}_1, \ldots, \mathbf{c}_n \rangle$ of n–tuples from $\mathcal{U}_{\mathcal{S}}$, $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{R}}$ respectively, such that $\langle \mathbf{a}_i, \mathbf{b}_i \rangle \in C$ and $\langle \mathbf{b}_i, \mathbf{c}_i \rangle \in \widehat{C}$ $(i = 1, \ldots, n)$ there is a relation $\widetilde{C} \in \mathfrak{Co}_{\mathcal{S},\mathcal{R}}$, such that $\widetilde{C} \supset \{\langle \mathbf{a}_i, \mathbf{c}_i \rangle \mid i = 1, \ldots, n\}$.
(i) $\mathfrak{F}$ is then called **transitive**, if for each pair $\mathcal{S} \xrightarrow{C} \mathcal{T}$ and $\mathcal{T} \xrightarrow{\widehat{C}} \mathcal{R}$ there is a relation $\mathcal{S} \xrightarrow{\widetilde{C}} \mathcal{R}$ in $\mathfrak{Co}_{\mathcal{S},\mathcal{R}}$, such that $\widetilde{C} \supset C \circ \widehat{C}$.

Now by showing that the corresponding property of frames holds on the canonical frame of a logic, we easily obtain e.g. the following completeness–results:

Theorem 5.7
The logic $FK + (NdI)$ is frame–complete wrt the class of all functional frames.

$FK + (NdD)$ *is frame–complete wrt the class of all injective frames.*
$FK + (NdE)$ *is frame–complete wrt the class of all existentially faithful frames.*
$FK + (CBF)$ *is frame–complete wrt the class of all existentially friendly frames.*
$FK + (NdF)$ *is frame–complete wrt the class of all fictionally faithful frames.*
$FK + (T)$ *is frame–complete wrt the class of all locally reflexive frames.*
$FK + (4)$ *is frame–complete wrt the class of all locally transitive frames.*

Definition 5.8 ((Semi–) categorial frames) A frame $\mathfrak{F}$ is said to be **semi–categorial**, if it is reflexive and transitive. Moreover, we say that $\mathfrak{F}$ is **categorial**, if $\mathfrak{F}$ is functional and semi–categorial.

As has already been noted, categorial frames that are also frames for classical predicate logic are semantically equivalent to $\mathcal{C}$–sets as e.g. defined in (Ghilardi 1991). That is to say, a formula is valid on all $\mathcal{C}$–sets if and only if it is valid on all categorial frames for classical predicate logic, i.e. where $U_w = D_w$ for all worlds w. Now it was shown in (Ghilardi 1992) that all (standard) quantified extensions of canonical propositional modal logics above **S4** are frame–complete wrt functor–semantics. Hence the semantics given in this paper allows for a wide class of frame–complete logics.

6 General Frames

In this last paragraph we want to sketch how to obtain general frame–completeness by enriching the notion of a modal frame by suitable algebras of 'admissible interpretations'. The situation is therefore quite analogous to the propositional case, where we have a natural possible worlds semantics, which is highly incomplete unless an algebraic component is added. The algebraic concept of a complex algebra as proposed below is in particular a straightforward extension of the concept of a boolean algebra with operators as known from propositional modal logic.

Definition 6.1 (Complex Algebras) Suppose $\mathfrak{F} = \langle \mathcal{U}, \mathcal{C} \rangle$ is a modal frame, where $\mathcal{U} = \{\langle U_w, D_w \rangle \ : \ w \in W\}$ is the associated family of universes. An **n–set** is a set of elements of the form $\langle \mathbf{a}, w \rangle$, where $w \in W$ and $\mathbf{a} \in (\mathcal{U}_w)^n$. If A is an n–set, put

$$\blacklozenge A := \{\langle \mathbf{b}, w \rangle \in (U_w)^n \times \{w\} \ : \ \text{there is a } \langle \mathbf{a}, v \rangle \in A \\ \text{and a } C \in \mathfrak{Co}_{w,v} \text{ such that } \langle \mathbf{b}, \mathbf{a} \rangle \in C\} \ .$$

If $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is a map and $\mathbf{a}$ an n–tupel, let $\sigma(\mathbf{a})$ denote the m–tupel $\langle a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(m)}\rangle$. In an m–set A and a map $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ are given, put

$$\widehat{\sigma}(A) := \{\langle \mathbf{a}, w\rangle \in (\mathcal{U}_w)^n \times \{w\} : \langle \sigma(\mathbf{a}), w\rangle \in A\} .$$

Furthermore, if $j \in \omega$, A is an n–set and $j \leq n$, define the operation $\mathbb{E}_j$ as follows:

$$\mathbb{E}_j(A) := \{\langle a_1, \ldots, a_{j-1}, c, a_{j+1}, \ldots, a_n, w\rangle \in (U_w)^n \times \{w\} \ : \ c \in U_w$$
$$\text{and there is a } b \in D_w, \text{ such that } \langle a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n, w\rangle \in A\}$$

Finally, if $i, j, n \in \omega$ with $i, j \leq n$ and A is an n–set, define the operation $id_{i,j}$ by

$$id_{i,j}(A) := \{\langle \mathbf{a}, w\rangle \in (U_w)^n \times \{w\} \ : \ \langle \mathbf{a}, w\rangle \in A \text{ and } a_i = a_j\}.$$

A **complex algebra** $\mathbb{G}_n$ **of type** n **based on** $\mathcal{U}$ is defined as a family of n–sets, which is closed under all boolean operations and also under the operations $\blacklozenge$, $\mathbb{E}_j$ for every $j \in \omega$ and also under $id_{i,j}$ for every pair $i, j \in \omega$. A **complex algebra** is then defined as a sequence $\mathbb{G} = \langle \mathbb{G}_n : n \in \omega\rangle$, in which $\mathbb{G}_n$ are complex algebras of type n $(i \in \omega)$ and $\bigcup_{i\in\omega} \mathbb{G}_i$ is also closed under the operation $\widehat{\sigma}$ for every σ.

Definition 6.2 (General Frames) A **general frame** $\gamma\mathfrak{F}$ is a pair $\langle \mathfrak{F}, \mathbb{G}\rangle$, where $\mathfrak{F}$ is a modal frame and $\mathbb{G}$ is a complex algebra based on $\mathfrak{F}$. An *interpretation* $\mathcal{I}$ in $\mathfrak{F}$ is called **admissible**, if for every n–place relation symbol P, $\mathcal{I}(P) \in \mathbb{G}_n$. A **model** based on a general frame $\langle \mathfrak{F}, \mathbb{G}\rangle$ is a triple $\langle \mathfrak{F}, \mathbb{G}, \mathcal{I}\rangle$, where $\mathcal{I}$ is an admissible interpretation. A formula is then said to be **valid** on a general frame $\gamma\mathfrak{F}$, if it holds in every model based on $\gamma\mathfrak{F}$.

The reader will immediately notice the close connection between these models and the cartesian metaframes of Shehtman and Skvortsov (1993) or the general cartesian metaframes of Shirasu (1998). The connections between these different types of models will be examined in full detail in a sequel to this paper.

Now, if a model $\langle \mathfrak{F}, \mathbb{G}, \mathcal{I}\rangle$ based on the general frame $\langle \mathfrak{F}, \mathbb{G}\rangle$ is given, we assign to each natural number n and each formula ϕ whose *free* variables are contained in $\bar{x} = \{x_1, \ldots, x_n\}$ and are exactly $\sigma(\bar{x}) = \langle x_{\sigma(1)}, \ldots, x_{\sigma(m)}\rangle$ (where $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$) an n–set $\widehat{\mathcal{I}}(\phi, n)$ inductively as follows.

$$\begin{array}{lcl}
\widehat{\mathcal{I}}(P^m(\sigma(\bar{x})), n) & := & \widehat{\sigma}(\mathcal{I}(P^m)) \\
\widehat{\mathcal{I}}(x_i \doteq x_j, n) & := & id_{i,j}(\mathcal{U}^n) \\
\widehat{\mathcal{I}}(\neg\phi, n) & := & \mathcal{U}^n - \widehat{\mathcal{I}}(\phi, n) \\
\widehat{\mathcal{I}}(\phi_1 \wedge \phi_2, n) & := & \widehat{\mathcal{I}}(\phi_1, n) \cap \widehat{\mathcal{I}}(\phi_2, n)
\end{array}$$

$$\begin{array}{lcl}\widehat{\mathcal{I}}(\Diamond\phi, n) & := & \blacklozenge\widehat{\mathcal{I}}(\phi, n) \\ \widehat{\mathcal{I}}(\exists x_j.\phi, n) & := & \mathbb{E}_j(\widehat{\mathcal{I}}(\phi, n))\end{array}$$

It follows that for every formula ϕ, whose free variables are contained in $\{x_1, \ldots, x_n\}$, $\widehat{\mathcal{I}}(\phi) \in \mathbb{G}_n$. The **canonical general $\mathcal{L}$–frame** is then defined as follows. The set of worlds and relations is as defined above. Likewise, $\mathcal{I}_{\mathcal{L}}$ is as previously defined on atomic predicates. We now define for every n a complex algebra of type n over $\mathcal{U}^n$ by means of the canonical interpretation as follows. Put

$$\mathbb{G}_n := \{\widehat{\mathcal{I}_{\mathcal{L}}}(\phi, n) : FV(\phi) \subseteq \{x_1, \ldots, x_n\}\} .$$

It is easily shown that this is a general frame, which we call the **canonical general frame**. Still, as might be observed, the theory of a general frame is *not always* closed under second–order substitutions and hence not a modal predicate logic in the usual sense. However, the theory of the canonical general frame is closed under these substitutions and this suffices to establish the following completeness result.

In the following theorem, an $\mathcal{L}$–frame is understood to be a modal frame $\mathfrak{F}$, such that $\mathfrak{F} \vDash \mathcal{L}$.

Theorem 6.3 (Completeness) *If $\mathcal{L}$ is a modal predicate logic and $\mathbb{K}$ is the class of all general $\mathcal{L}$–frames, then $\mathcal{L}$ is sound and complete wrt the class $\mathbb{K}$, i.e. $\mathcal{L} = \bigcap_{\gamma\mathfrak{F}\in\mathbb{K}} Th(\gamma\mathfrak{F})$.*

Proof. Let $\gamma\mathfrak{F}_{\mathcal{L}}$ be the canonical general frame and suppose that there is a $\phi \in \mathcal{L}$, such that $\gamma\mathfrak{F}_{\mathcal{L}} \nvDash \phi$. Then there is an admissible interpretation $\mathcal{I}$, a possible world $\mathcal{S}_\Delta$ and a valuation β such that $\langle \mathcal{S}_\Delta, \mathcal{I}, \beta\rangle \nvDash \phi$. Because $\mathcal{L}$ is closed under first–order substitutions we can wlog assume that β is the canonical valuation υ_Δ. Let $P_1^{n_1}, \ldots, P_m^{n_m}$ be all the relation symbols occurring in ϕ. Since $\mathcal{I}$ is admissible, $\mathcal{I}(P_i^{n_i}, \Delta) = \widehat{\mathcal{I}_{\mathcal{L}}}(\psi_i) \in \mathbb{G}_{n_i}$ for $i = 1, \ldots, m$. In particular *fvar*$(\psi_i) \subseteq \{x_1, \ldots, x_{n_i}\}$. Since $\phi \in \mathcal{L}$, it follows by closure under second–order substitutions that $\widetilde{\phi} = (\psi_1/P_1^{n_1}, \ldots, \psi_m/P_m^{n_m})\phi \in \mathcal{L}$ as well. But since $\mathcal{I}(P_i^{n_i}) = \widehat{\mathcal{I}_{\mathcal{L}}}(\psi_i)$ it follows by induction that $\langle \mathcal{S}_\Delta, \mathcal{I}_{\mathcal{L}}\rangle \nvDash \widetilde{\phi}$, which contradicts $Th(\mathfrak{f}_{\mathcal{L}}) = \mathcal{L}$. This shows that the canonical general frame $\gamma\mathfrak{F}$ is an $\mathcal{L}$–frame. Now, completeness follows immediately by appealing to the Canonical Model Theorem. $\dashv$

7 Conclusions and Further Work

We have introduced a semantics that combines the mathematical generality of functor semantics, metaframes or hyperdoctrines with aspects of Counterpart Theory. By constructing a canonical model for each model predicate logic $\mathcal{L}$, we have shown completeness wrt modal structures. Furthermore, by defining appropriate conditions on Counterpart relations, many interesting frame–completeness results are easily established. By enriching the notion of a modal frame by a suitable complex algebra of 'admissible interpretations', general frame–completeness is obtained. Finally, the use of free logic does not only allow for an easy refutation of formulae like the Converse Barcan Formula, it is also the starting point for a proper treatment of non–denoting singular terms, talk of fictional objects or an investigation of different theories of definite descriptions in modal contexts.

Since we have not considered these last topics in detail here, this is one of the subjects to be investigated in more detail in the future. Also, the exact connection between the different proposed semantics is to be spelled out. It should also be investigated how the semantics works with richer modal languages that use—besides an existence predicate—e.g. an actuality operator, universal modalities or lambda abstraction as discussed in (Fitting and Mendelsohn 1998). As a last point in case we would like to point to the problem of investigating the question, which (philosophical) notion of object or individual is presupposed by the kind of semantics we have discussed in this paper. We believe that modal predicate logic in itself does not fix the notion of an object uniquely, since very different semantics involving quite distinct notions of an individual can give rise to general completeness results.

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Marcus Kracht
Freie Universität Berlin
II. Mathematisches Institut
E-mail: `kracht@math.fu-berlin.de`

Oliver Kutz
Universität Leipzig
Institut für Informatik
E-mail: `kutz@informatik.uni-leipzig.de`