Invariant Logics

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Abstract. A modal logic Λ is called *invariant* if for all automorphisms α of NExt $\mathbf{K}$, $\alpha(\Lambda) = \Lambda$. An invariant logic is therefore uniquely determined by its surrounding in the lattice. It will be established among other that all extensions of $\mathbf{K.alt}_1$, $\mathbf{S5}$ and $\mathbf{G.3}$ are invariant logics. Apart from the results that are being obtained, this work contributes to the understanding of the combinatorics of finite frames in general, something wich has not been done except for transitive frames. Certain useful concepts will be established, such as the notion of a *d–homogeneous frame*.

1. Introduction

In [3] we have investigated the groups of automorphisms of various lattices of modal logics. A full description of $\mathfrak{Aut}(\mathrm{NExt}\,\mathbf{S4.3})$ and $\mathfrak{Aut}(\mathrm{NExt}\,\mathbf{K.alt_1})$ was obtained. Moreover, we have shown that every automorphism of NExt $\mathbf{S4}$ fixes all logics with finite model property, and so also all logics of finite codimension. We raised the question about the group of automorphisms of NExt $\mathbf{K}$. Here, we shall make a modest step towards the structure of this group. We shall study logics which are invariant under all automorphisms of NExt $\mathbf{K}$. Of course, knowing which logics must be left invariant under any automorphism helps in this direction because it narrows down the possible actions of an automorphism. It is our belief that all logics with the finite model property are invariant. This narrows down the choice of automorphisms substantially, though it is still conceivable that there are continuously many autormophisms.

2. Basic Terminology and Notation

We follow the notation and terminology of [2] for the basic terminology of modal logic and [3] for the notions relevant to automorphisms of lattices of modal logics. We will quickly review the basic instruments which we will need for our purposes. Recall that the normal logics form a distributive lattice (in fact a locale), which is denoted by NExt $\mathbf{K}$. The lattice operations are denoted by $\sqcup$ and $\sqcap$ (and the infinitary operations by $\bigsqcup$ and $\bigsqcap$). Notice that $\Theta \sqcap \Lambda = \Theta \cap \Lambda$, although not necessarily $\Theta \sqcup \Lambda = \Theta \cup \Lambda$.

Definition 2.1. *A modal logic Λ is said to be (**absolutely**) **invariant**, if for all automorphisms α of NExt $\boldsymbol{K}$, $\alpha(\Lambda) = \Lambda$.*

A **Kripke–frame** is a pair $\mathfrak{F} = \langle F, \lhd\rangle$, where F is a set (possibly empty) and $\lhd \subseteq F^2$ a binary relation. If $G \subseteq F$ then $\mathfrak{F} \upharpoonright G := \langle G, \lhd \cap G^2\rangle$ is the induced subframe on G. A frame isomorphic to some $\mathfrak{F} \upharpoonright G$ is called a **subframe** of $\mathfrak{F}$. We write $\mathfrak{G} \subseteq \mathfrak{F}$ if $\mathfrak{G}$ is a subframe of $\mathfrak{F}$. Special subframes are the generated subframes, which are based on *generated subsets*. $S \subseteq F$ is a **generated subset** of $\mathfrak{F}$ if for all $x \in S$ and $x \lhd y$ also $y \in S$. We write $\mathfrak{F} \uparrow x$ for the generated subframe induced by x. It is based on the least generated subset containing x. $x \in F$ is a **root** of $\mathfrak{F}$ if the least generated subset containing x is F. $\mathfrak{F}$ is called **rooted** if it has a root. Given a Kripke–frame $\mathfrak{F}$, put $\mathsf{Th}\,\mathfrak{F} := \{\varphi : \mathfrak{F} \models \varphi\}$. Finally, let $\mathfrak{F} \leq \mathfrak{G}$ iff $\mathsf{Th}\,\mathfrak{F} \leq \mathsf{Th}\,\mathfrak{G}$. This is a converse well–quasi–ordering: it is transitive, and contains no infinite strictly upgoing chain. However, we can have $\mathfrak{F} \leq \mathfrak{G} \leq \mathfrak{F}$ and nevertheless $\mathfrak{F} \neq \mathfrak{G}$, even $\mathfrak{F} \not\cong \mathfrak{G}$! One example is $\mathfrak{F} \oplus \mathfrak{F}$, the disjoint sum of $\mathfrak{F}$ with itself. Of particular interest in this paper are the logics of rooted finite Kripke–frames. We therefore define $\mathbb{I}$ to be the set of all $\mathsf{Th}\,\mathfrak{F}$, $\mathfrak{F}$ rooted and finite. As is well–known ([1]), $\mathbb{I}$ does not coincide with

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the set of intersectively irreducible logics of finite codimension. $\mathbb{I}$ is ordered by $\leq$. We shall often identify a logic of a frame with the frame itself, if that carries no risk of confusion. In that sense we may for example think of an automorphism of NExt $\mathbf{K}$ as inducing an automorphism of the set of all rooted finite Kripke–frames.

A logic Λ is **prime** in NExt $\mathbf{K}$ if whenever $\bigcap_{i \in I} \Theta_i \leq \Lambda$, there is an $i \in I$ such that $\Theta_i \leq \Lambda$. [1] has shown that Λ is prime in NExt $\mathbf{K}$ iff $\Lambda = \mathsf{Th}\,\mathfrak{F}$ for some finite, rooted, cycle–free frame. Let us denote the se of prime logics by $\mathbb{P}$. Then we have the following:

Lemma 2.2. *Every automorphism of NExt* $\boldsymbol{K}$ *fixes the set* $\mathbb{P}$.

It does *not* follow that each membed is fixed by a given automorphism. In fact, establishing this for a small class of frames requires a lot of work. We shall briefly mention the following important fact.

Lemma 2.3. *Let* $\Lambda \in \mathbb{P}$ *and* $\Theta \geq \Lambda$. *Then* Θ *is tabular. Moreover, it is* $\sqcap$*–irreducible iff it is in* $\mathbb{P}$.

Further, notice the following:

Lemma 2.4. $\mathbb{P}$ *is an upper subset of* $\langle \mathbb{I}, \leq \rangle$.

This means that if $\Theta \in \mathbb{P}$ and $\Lambda \geq \Theta$ is a member of $\mathbb{I}$, then $\Lambda \in \mathbb{P}$ as well. It follows from Lemma 2.3 that given $\mathfrak{F} \in \mathbb{P}$, there is an upper bound on the number of logics containing $\mathsf{Th}\,\mathfrak{F}$, which can be uniformly determined on the basis of $|F|$.

Define $d_n := \Box^n \bot \wedge \Diamond^{n-1} \top$, $n > 0$. Say that a point x in a cycle–free frame $\mathfrak{F}$ is of **depth** n if $x \models d_n$. It is not hard to show that any point in $\mathfrak{F}$ has a depth and that this depth is unique. The depth of $\mathfrak{F}$ is the largest number n such that there exists a node of depth n.

In Rautenberg [5], Page 232, one finds a picture of the upper part of NExt $\mathbf{K}$. We shall be dealing only with tabular irreducible logics. First, there are two coatoms, by a theorem of Makinson ([4]). Then follow countably many logics of depth 2. These are among other the logics of the loops of prime order. Let $\mathfrak{Loop}_n := \langle \{0, 1, \ldots, n-1\}, \lhd \rangle$, where $j \lhd j$ iff $j \equiv i + 1 \pmod{n}$. Then the logic $\mathsf{Th}\,\mathfrak{Loop}_n$ has codimension 2 exactly when n is a prime number (see [2]). There are many more logics of depth 2, but this may suffice for an indication of the complexity of the lattice, even when we restrict the attention to tabular logics.

In order to establish that some logic Λ is invariant under automorphisms it is enough to find a formula $\varphi(x)$ in some logical language using the signature of lattices such that $\{\Lambda\} = \{\Theta : \text{NExt}\,\mathbf{K} \models \varphi(\Theta)\}$. In sequel it is enough to use monadic second order logic (MSO). The nonlogical symbols will be $\leq$, $\sqcap$ and $\sqcup$. Of course, some higher order language might also be used but is not necessary for our purposes. However, we shall make use of the following fact.

Lemma 2.5. *The set of invarint logics is closed under* $\sqcap$, $\sqcup$, $\bigsqcap$ *and* $\bigsqcup$. *In particular, it forms a sublocale of NExt* $\boldsymbol{K}$.

The supremum and infimum of infinite sets is defined as follows:

$$\begin{array}{rcl} \mathit{inf}(P,x) & := & (\forall y)(y \in P \rightarrow x \leq y) \wedge (\forall z)((\forall y)(y \in P \rightarrow y \leq z) \rightarrow z \leq x) \\ \mathit{sup}(P,x) & := & (\forall y)(y \in P \rightarrow x \geq y) \wedge (\forall z)((\forall y)(y \in P \rightarrow y \geq z) \rightarrow z \geq x) \end{array}$$

Definition 2.6. *Let* $S \subseteq$ *NExt* $\boldsymbol{K}$. S *is called* ***MSO–definable*** *if there is an MSO–formula* φ *such that* $S = \{\Theta : \textit{NExt}\,\boldsymbol{K} \models \varphi(\Theta)\}$. *A logic* $\Lambda \in$ *NExt* $\boldsymbol{K}$ *is* ***MSO–definable*** *if* $\{\Lambda\}$ *is MSO–definable. Throughout this paper we shall use* ***definable*** *instead of* ***MSO–definable***.

It is worthwile to establish the definability of some basic sets of logics.

Lemma 2.7. *The set of* $\sqcap$*–irreducible logics and the set of prime logics are definable.*

Proof. The defining formulae are

$$\begin{array}{rcl} \varphi_i(x) & := & (\forall yz)(x \doteq y \sqcap z \rightarrow (y \doteq x \vee z \doteq x)) \\ \varphi_p(x) & := & (\forall P)((\forall y)(\mathit{inf}(P,y) \rightarrow y \leq x) \rightarrow (\exists y)(y \in P \wedge y \leq x)) \end{array}$$

□

Lemma 2.8. *Suppose that S is definable. Then for every automorphism α of NExt $\boldsymbol{K}$, $\alpha[S] = S$. It follows that $\bigsqcap S$ and $\bigsqcup S$ are invariant. In particular, if Λ is definable, it is invariant.*

This proves once more the observation 2.2. Notice that since there are only countably many formulae, not every subset is MSO–definable. Nevertheless, it may very well be that all subsets are invariant (namely when there exists only one automorphism), so the converse of the previous theorem is false. We shall note however that α is the identity already when it fixes every compact (= finitely axiomatizable) logic. Since the set of compact logics is definable, we can in principle determine the action of an automorphism α on its action on a countable definable subset of NExt $\mathbf{K}$.

3. Inherently 1–covered logics

We shall identify $\mathbb{P}$ in some canonical way with a set of finite rooted Kripke–frames, which we will also call $\mathbb{P}$. $\mathbb{P}$ is ordered by $\leq$, as defined above. It is an upper subset of $\langle \mathbb{I}, \leq \rangle$.

Definition 3.1. *Suppose that $\mathfrak{F} \in \mathbb{P}$. We say that $\mathfrak{F}$ is* ***n–covered*** *if it has at most n covers in $\mathbb{I}$ (iff it has at most n covers in $\mathbb{P}$). $\mathfrak{F}$ is* ***inherently n–covered*** *if every $\mathfrak{G} \geq \mathfrak{F}$ is n–covered. We denote by $\mathbb{C}^n$ the set of all inherently n–covered cycle-free Kripke–frames (and the logics thereof). Finally, let*

$$\Gamma^n := \mathsf{Th}\,\mathbb{C}^n$$

Clearly, these notions are definable. From Lemma 2.8 we infer

Lemma 3.2. *Let $\alpha \in \mathfrak{Aut}(\text{NExt}\,\boldsymbol{K})$. Then $\alpha(\Gamma^n) = \Gamma^n$.*

Of particular interest in the study are the logics in $\mathbb{C}^1$. Obviously, they represent the easiest logic of this kind. Here is a useful way to identify these logics.

Lemma 3.3. *Let Λ be prime. Λ is in $\mathbb{C}^1$ iff NExt Λ is finite and linear.*

It does not follow, of course, that the generating frame of Λ is linear; neither does it follow that NExt Γ^1 is linear. To the contrary, this is a quite complex lattice, whose structure we will unravel to some extent. We remark here the following fact.

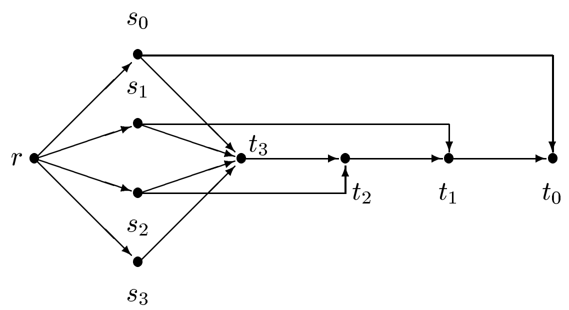
Proposition 3.4. *$\boldsymbol{K} = \bigsqcup_{n\in\omega} \Gamma^n$. Furthermore, $\Gamma^{n+1} < \Gamma^n$ for all $n \in \omega$.*

Proof. First, for any finite Kripke–frame there is an n such that it is inherently n–covered. So, the first claim follows. For the second, consider the following frame $\mathfrak{L}_n := \langle L_n, \lhd_n \rangle$ defined by

$$L_n \;:=\; \{r\} \cup \{s_i : i < n\} \cup \{t_i : i < n\}$$

$$\lhd_n \;:=\; \left\{ \begin{array}{ll} & \{\langle r, s_i\rangle : i < n\} \\ \cup & \{\langle t_j, t_{j-1}\rangle : 0 < j < n\} \\ \cup & \{\langle s_i, t_{n-1}\rangle : i < n\} \\ \cup & \{\langle s_i, t_i\rangle : i < n\} \end{array} \right.$$

The picture below show the frame $\mathfrak{L}_4$.



It is easily established that $\mathfrak{L}_n$ has no contraction images other than $\mathfrak{L}_n$. (A contraction must be depth preserving, and so it can only collapse some s–points. But this is impossible, as they have different successors.) The subframes generated by the s_i are therefore the only immediate covers of $\mathfrak{L}_n$. They are all non–isomorphic, showing that $\mathfrak{L}_n$ has n covers. ☐

Before we consider the structure of $\mathbb{C}^1$–frames, let us specialize a little further.

Definition 3.5. *Let $\mathfrak{F} \in \mathbb{P}$. Then $\mathfrak{F}$ is called **linear** if for any $n \in \omega$ there is at most one point of depth n in $\mathfrak{F}$.*

(For a general notion of linearity, see Section 6.) So far we have not established any tool to identify logics of linear frames. However, a useful thing to note is the following:

Lemma 3.6. *Let $\mathfrak{F}$ be linear. Then any contraction image of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}$. Furthermore, the codimension of $\mathsf{Th}\,\mathfrak{F}$ equals the number of points of $\mathfrak{F}$.*

Proof. The first claim is easy to verify. Now suppose that $\mathfrak{G} \geq \mathfrak{F}$. Then $\mathfrak{G}$ is isomorphic to a generated subframe of $\mathfrak{F}$. Now suppose that the root of $\mathfrak{F}$ has depth n and let y be of depth $n-1$. Then y generates a frame $\mathfrak{G}$ of depth $n-1$. By linearity, $|G| = |F| - 1$. This shows that NExt $\mathsf{Th}\,\mathfrak{F}$ has $n+1$ points, and so $\mathsf{Th}\,\mathfrak{F}$ has codimension n. ☐

Definition 3.7. *Suppose that $\mathfrak{F}$ is cycle–free. $\mathfrak{F}$ is called **d–homogeneous** if for any pair x, y of points of equal depth the following holds: if $x \lhd u$ and u has depth k, then there is a v of depth k such that $y \lhd v$.*

Obviously, linear frames are d–homogeneous.

Lemma 3.8. *Suppose that $\mathfrak{F} \in \mathbb{P}$ is d–homogeneous. Then there exists a (uniquely determined) linear frame $\mathfrak{G}$ and a p–morphism $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$. Furthermore, if $\mathfrak{F}$ is contractible to a linear frame, $\mathfrak{F}$ is d–homogeneous.*

Proof. Put $G := \{n :\ \text{exists } x \in F : x \models d_n\}$, and let $m \blacktriangleleft n$ iff there exists x of depth m and y of depth n such that $x \lhd y$. $\mathfrak{G} := \langle G, \blacktriangleleft \rangle$. Now put $\pi(x) = n$ if x has depth n. It is easy to check that this is a p–morphism. The uniqueness follows from the fact that if x is of depth n in $\mathfrak{F}$, it must be of depth n in $\mathfrak{G}$. So $\mathfrak{G}$ is the smallest frame onto which a p–morphism exists, and it is easily seen that there exist no p–morphism between linear frames. Hence, $\mathfrak{G}$ exists and is unique. Now assume that $\mathfrak{F}$ is contractible to a linear frame, say $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$. Then x and y have the same depth iff $\pi(x) = \pi(y)$. So, assume that $x \lhd u$ and let y be of same depth as x. Then $\pi(x) = \pi(y)$ and therefore $\pi(y) \lhd \pi(u)$. It follows that there is a v such that $y \lhd v$ and $\pi(v) = \pi(u)$. So, v has the same depth as u. ☐

So, $\mathfrak{F}$ is d–homogeneous exactly if it is contractible to a linear frame. For $\mathbb{P}$–frames in general we shall define the concept of a *depth profile*.

Definition 3.9. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame. Then the **depth profile** of $\mathfrak{F}$, $\wp(\mathfrak{F})$, is the set of pairs of numbers $\langle m, n \rangle$ such that there exists a point of depth m seing a point of depth n. For a point x, the **depth profile** of x, $\wp(x)$, is the set of all numbers n such that there exists a successor of x of depth n.*

Clearly, a linear frame is uniquely characterized by its depth profile. We may also note the following, which is immediate from the definitions.

Lemma 3.10. *Let $\mathfrak{F}$ be d–homogeneous, x of depth m and $\langle m, n \rangle \in \wp(\mathfrak{F})$. Then there exists a y of depth n such that $x \lhd y$.*

Lemma 3.11. *Let $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ a p–morphism. Then*

1. *$\mathfrak{F}$ is d–homogeneous iff $\mathfrak{G}$ is.*
2. *$\wp(\mathfrak{F}) = \wp(\mathfrak{G})$.*

Proof. The first claim follows from Lemma 3.8. For the second notice that a p–morphism is depth preserving. □

We define for a given $\mathfrak{F}$ the **order depth** the length of a longest ascending chain of $\sqcap$–irreducible logics. This is by definition a lattice invariant. (It is *not* the same as the codimension of that logic.)

Lemma 3.12. *Let $\mathfrak{F} \in \mathbb{P}$ be d–homogeneous. Then the order depth of $\mathfrak{F}$ coincides with $|F|$.*

Proof. We have shown this already for linear frames. So, assume that $\mathfrak{F}$ is not linear. Choose x and y of same depth such that x and y see the same set of points. Such a pair exists. For let m be the least number such that there is more than one point of depth m. Since $\mathfrak{F}$ is not linear, m exists. Choose x and y of depth m. Then x and y see the same set of points, as is easy to see. (For otherwise x and y generate nonisomorphic linear frame $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$.) Now collapse x and y into the same point and call the resulting frame $\mathfrak{G}$. Then $\mathfrak{G} \in \mathbb{P}$ and $|G| = |F| - 1$. Now an easy induction on $|F|$ establishes the claim. □

As it turns out, $\mathbb{C}^1$–frames are d–homogeneous:

Lemma 3.13. *Let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame. Then $\mathfrak{F}$ is d–homogeneous.*

Proof. Suppose the claim is false. Then there exists a smallest $\mathbb{C}^1$–frame $\mathfrak{F}$ which is not d–homogeneous. This means that there exist points x and y of identical depth, say n, such that x has a successor of depth $m < n$, but y does not. Consider the subframes $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$ generated by x and y. By choice of $\mathfrak{F}$ they are d–homogeneous and therefore possess a linear contraction image $\mathfrak{G}_x$ and $\mathfrak{G}_y$, respectively. By choice of x and y, $\mathfrak{G}_x$ and $\mathfrak{G}_y$ have the same depth but are not isomorphic. Hence we have neither $\mathfrak{G}_x \geq \mathfrak{G}_y$ nor $\mathfrak{G}_x \leq \mathfrak{G}_y$. But this is impossible by Lemma 3.3. □

As a consequence we note that, by Lemma 3.12, the order depth of a $\mathbb{C}^1$–frame $\mathfrak{F}$ is exactly $|F|$. So we can reconstruct the number of elements in such a frame just from the lattice of its extensions. A cycle–free frame is of depth n if no point has depth $> n$ and there is at least one point of depth n. Now, rooted frames of depth 2 have a rather simple structure. They consist of a single root and a number of depth 1 successors. Let $\mathfrak{F}$ be of depth 2 and let $\mathfrak{F}$ have $n+1$ elements; then we denote $\mathfrak{F}$ by $\mathfrak{B}_n$.

Lemma 3.14. *Let $n > 1$. $\mathfrak{B}_n$ has exactly one $\mathbb{C}^1$–cocover, namely $\mathfrak{B}_{n+1}$.*

Proof. Suppose that $\mathfrak{F}$ is a cocover of $\mathfrak{B}_n$ and assume that $\mathfrak{F} \in \mathbb{C}^1$. Then $\mathfrak{F}$ cannot be of depth 3, since then it has several cocovers (namely, there is a generated subframe of depth 2 isomorphic to $\mathfrak{B}_n$ and a contraction image, which has $n-1$ points of depth 1). So it is of depth 2, and hence isomorphic to $\mathfrak{B}_m$ for some m. It is easy to see that $m = n+1$. This shows that $\mathfrak{B}_n$ has exactly one cocover in $\mathbb{C}^1$. □

Now, there are three non–isomorphic rooted, cycle–free frames of cardinality 3, two of which are linear. The third is $\mathfrak{B}_2$. We can define $\mathfrak{B}_2$ since the other two frames have too many $\mathbb{C}^1$–cocovers. For it follows from the next lemma that the linear frames with three elements have no less than 4 $\mathbb{C}^1$–cocovers.

Lemma 3.15. *Let $\mathfrak{F}$ be rooted and let $|F| = n$. Then $\mathfrak{F}$ has at most 2^{n-1} linear cocovers. If $\mathfrak{F}$ is linear, this bound is exact.*

Proof. The new point is the new root. Its set of successors is any set containing at least the root of $\mathfrak{F}$. There are 2^{n-1} such sets. Some cocovers may be isomorphic, but if $\mathfrak{F}$ is linear, they are all linear as well and non isomorphic. □

In order to make progress in distinguishing various frames of identical size we shall develop a formula that allows us to count the number of cocovers. The idea is as follows. We have already established how many $\mathbb{C}^1$–frames exist of which $\mathfrak{F}$ is a maximal generated subframe, now we shall establish how many $\mathbb{C}^1$–frames exist that cover $\mathfrak{F}$ and can be mapped onto it. By previous theorems,

$\mathfrak{G}$ cocovers $\mathfrak{F}$ means that $\mathfrak{G}$ has one more point than $\mathfrak{F}$. So let $\pi : G \to F$ be a contraction and x and y different points such that $\pi(x) = \pi(y)$. (By our assumptions, $\{x, y\}$ is unique.) Then the set of successors of x and y in $\mathfrak{G}$ coincide, by the p–morphism conditions. Furthermore, the set P of predecessors of $\pi(x)$ in $\mathfrak{F}$ is the union of two sets $\pi[X]$ and $\pi[Y]$, where X is the set of predecessors of x in $\mathfrak{G}$, and Y the set of predecessors of y in $\mathfrak{G}$. Both X and Y are nonempty, since $\mathfrak{G}$ is rooted. Conversely, if in $\mathfrak{G}$ we find two points x and y such that their set of successors is identical, then (and only then) is the collapsing of x and y a p–morphism, and the set P of predecessors of $\pi(x)$ is the union of (the π-image of) the set of predecessors of x and that of y. So, and cocover is obtained in this way. Obviously, an upper bound on the number of cocovers of which $\mathfrak{F}$ is a p–morphic image is obtained by taking all possible points x distinct from the root, and all possible representations of $P := \{y : y \lhd x\}$ as unions of nonempty subsets. This bound can be very high indeed. Notice also that it is only an upper bound, since some cocovers may turn out to be isomorphic.

There is an immediate application. Let $\mathfrak{Ch}_n$ be defined by

$$\begin{array}{rcl} \mathfrak{Ch}_n & := & \langle \{0, 1, \ldots, n-1\}, \lhd_n \rangle \\ \lhd_n & := & \{\langle i, i-1 \rangle : 0 < i < n\} \end{array}$$

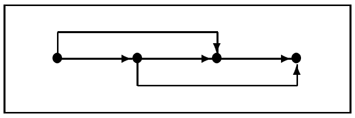
This chain has, by the above theorems, $2^{n-1} + n - 1$ $\mathbb{C}^1$–cocovers. For either $\mathfrak{Ch}_n \rightarrowtail \mathfrak{F}$, in which case $\mathfrak{F}$ is linear and there are 2^{n-1} nonisomorphic choices. But if $\mathfrak{G} \twoheadrightarrow \mathfrak{Ch}_n$, then $\mathfrak{G}$ is obtained by adding a point of depth $< n$. For each given depth there exists only one possibility, as is easily seen. Hence there are $n - 1$ nonisomorphic choices.

Now let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame. If it is linear it has exponentially many linear cocovers. We shall now show that if it is not linear, there are at most linearly many.

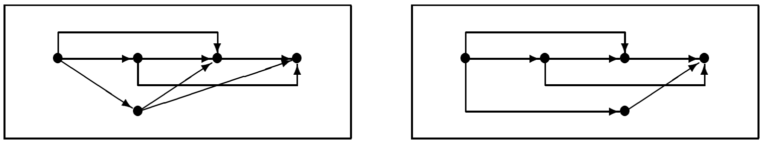
Lemma 3.16. *Let $\mathfrak{L}$ be linear and x the root. Then there are card$(\wp(x))$ many nonlinear $\mathbb{C}^1$–cocovers of $\mathfrak{L}$, which are all inherently 1–covered.*

Proof. Assume that $\mathfrak{L}$ has depth d with root x. Let $\mathfrak{F}$ be a d–homogeneous cocover of $\mathfrak{L}$. By Lemma 3.12, $\mathfrak{F}$ has one point more than $\mathfrak{L}$ itself. Further, $\mathfrak{F}$ has the same depth profile as $\mathfrak{L}$, by Lemma 3.11. This means that we can consider $\mathfrak{L}$ to be a subframe of $\mathfrak{F}$. So, we now assume that $L \subseteq F$. Let $F - L = \{y\}$. Let y be of depth k. It is easy to see that we must have $v \lhd y$ iff $v = x$. For otherwise let u be the successor of x of depth $d - 1$. Then $\mathfrak{F} \uparrow u$ is a cover of $\mathfrak{F}$ not isomorphic to $\mathfrak{L}$ (since it contains y). Consequently, $k \in \wp(x)$. Second, let w be of depth k, $w \neq y$. Since $\wp(\mathfrak{F}) = \wp(\mathfrak{L})$, we must have $\wp(y) = \wp(w)$, which is to say that w and y have the same successors. So, $\mathfrak{F}$ is uniquely determined by the number k. On the other hand, for each $k \in \wp(\mathfrak{L})$, such a frame exists. ☐

We denote the frames defined in the proof by $\mathfrak{L}_k$ (see also below for the notation). For example, the following frame



has these two nonlinear $\mathbb{C}^1$–cocovers:



In order to discriminate the nonlinear cocovers from the linear cocovers, we shall investigate the structure of $\mathbb{C}^1$–frames in more detail.

Definition 3.17. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame. We say that $\mathfrak{F}$ is of d–**width** m iff there are exactly m points of depth d. The **width** of $\mathfrak{F}$ is the maximum of all d–widths, $d \in \omega$.*

Lemma 3.18. *Let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame. Suppose that $\mathfrak{F}$ is of d–width 4. Then for all $d' < d$, $\mathfrak{F}$ is of d'–width 1.*

Proof. Let u_i, $i < 3$, be the points of depth d. Assume that there are two points of depth d', where $d' < d$. By applying suitable p–morphisms we can arrange it that there is no $d'' \neq d'$ and $d'' < d$ such that $\mathfrak{F}$ is d''–width < 1. Call the points of depth d' x and y. (Case 1.) $\langle d, d'\rangle \notin \wp(\mathfrak{F})$. Then the contraction of u_0 and u_1 into one point is a p–morphism, and the contraction of x and y into one point. So, $\mathfrak{F}$ has two covers. Hence this case does not arise. (Case 2.) $\langle d, d'\rangle \in \wp(\mathfrak{F})$. Then each u_i sees at least x or y. As $\{x, y\}$ has only three nonempty subsets, there are i and j such that $i < j$ and u_i and u_j have the same set of successors of depth d'. It follows that they have the same set of successors of any depth $< d$. Hence, contracting u_i and u_j to a single point is a p–morphism, and so $\mathfrak{F}$ has two nonisomorphic covers. So this case does not arise either. We have reached a contradiction and the claim is proved. □

Lemma 3.19. *Suppose that $\mathfrak{F}$ is a nonlinear $\mathbb{C}^1$–frame. Then there is no infinite, homogeneously 2–branching tree of $\mathbb{C}^1$–frames rooted at $\mathfrak{F}$.*

Proof. If $\mathfrak{F}$ is nonlinear and $\mathfrak{G}$ a $\mathbb{C}^1$–cocover of $\mathfrak{G}$, then $\mathfrak{G}$ has the same depth as $\mathfrak{F}$. Let it be d. Consider the number of frames which have the same order depth κ, where $\kappa > 3d - 2$. These frames cannot be of width < 4. Hence there is some d' such that they are of d'–width ≥ 4. By Lemma 3.18, $\mathfrak{F}$ is of d''–width < 4 for all $d'' \neq d'$. The number of such frames is bounded by some number just depending on d. This shows that no homogeneously 2–branching tree of $\mathbb{C}^1$–frames rooted at $\mathfrak{F}$ can exist. □

Lemma 3.20. *Let $\mathfrak{L}$ be linear and of depth > 1. Then there is an infinite, homogeneously 2–branching tree of $\mathbb{C}^1$–frames rooted at $\mathfrak{F}$.*

This follows basically from Lemma 3.15 and the fact that we are dealing with $\mathbb{C}^1$–frames.

Lemma 3.21. *The class $\mathbb{L}$ of linear frames is definable.*

Proof. There is an MSO–formula $t(x)$ which is true of Θ iff Θ is the root of some homogeneously 2–branching tree of $\mathbb{C}^1$–frames. Now consider the following formula:

$$\ell(x) := \mathbb{C}^1(x) \wedge (\forall y)(y \geq x \to t(x))$$

We claim that Θ is the logic of a linear frame iff $\ell(\Theta)$. But this is easily shown by induction on the order depth of Θ. □

Here is now a first result:

Lemma 3.22. *A linear frame is isomorphic to some $\mathfrak{Ch}_n$ iff its unique nonlinear $\mathbb{C}^1$–cocover has exactly one $\mathfrak{C}^1$–cocover.*

It follows that the set of logics $\{\mathsf{Th}\,\mathfrak{Ch}_n : n \in \omega\}$ is definable. Recall that $\mathbf{K.alt}_n$ is the logic of all frames with the property that a given point has at most n immediate successors. Then we get the following result.

Theorem 3.23. *$\boldsymbol{K.alt}_1$ is invariant.*

Definition 3.24. $\mathbb{L}$ *denotes the set of linear $\mathbb{P}$–frames, $\mathbb{D}$ the set of d–homogeneous $\mathbb{P}$–frames.*

Lemma 3.25. *$\mathfrak{F}$ is rooted and d–homogeneous iff for any two linear frames $\mathfrak{G}_1, \mathfrak{G}_2 \geq \mathfrak{F}$ we have $\mathfrak{G}_1 \geq \mathfrak{G}_2$ or $\mathfrak{G}_2 \geq \mathfrak{G}_1$.*

Proof. Suppose first that $\mathfrak{F}$ is d–homogeneous. Let $\mathfrak{G}_1, \mathfrak{G}_2 \geq \mathfrak{F}$. We can assume that they are images of some generated subframes $\mathfrak{H}_1, \mathfrak{H}_2 \subseteq \mathfrak{F}$. Let $\pi_1 : \mathfrak{G}_1 \twoheadrightarrow \mathfrak{H}_1$ and $\pi : \mathfrak{G}_2 \twoheadrightarrow \mathfrak{H}_2$. Assume that $|G_1| \leq |G_2|$. Consider the map $\rho : G_1 \to G_2$ which assigns to each point of depth k in $\mathfrak{G}_1$ the unique point of depth k in $\mathfrak{G}_2$. We claim that this is a p–morphism. So, assume that x is of depth k in $\mathfrak{G}_1$ and y $(= \rho(x))$ is of depth k in $\mathfrak{G}_2$. Then there are $u, v \in F$ such that $\pi_1(u) = x$ and $\pi_2(v) = y$. Moreover, u and v have the same depth. Suppose now that $x \lhd x'$ in $\mathfrak{G}_1$. Then there is a $u' \in F$ such that $x' \lhd u'$. Since $\mathfrak{F}$ is d–homogeneous, there is a v' of the same depth as u' such that $y' \lhd v'$. Hence, as $v' \in \mathfrak{H}_2$, $y \lhd \pi_2(v')$, and $v := \pi_2(v')$ has the same depth as v'. By definition, $v = \rho(u)$. The argumentation can be reversed. This shows that ρ is a p–morphism. Now suppose that $\mathfrak{F}$ is not d–homogeneous. Then there are $x, y \in F$ or identical depth and a u of depth k such that $x \lhd u$ but no v of depth k exists such that $y \lhd v$. We choose x of minimal depth with this property. Consider $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$. We claim that these frames are d–homogeneous. For example, if $u, v \in \mathfrak{F} \uparrow x$ and the depth of the two points is less than the depth of x, by minimality of x, u has a successor of depth n iff v has. However, there is exactly one point of depth equal to the depth of x. So, the claim is proved. It follows that $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$ can be mapped onto some linear frames $\mathfrak{L}_1$ and $\mathfrak{L}_2$. By choice of x and y, $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are not isomorphic, but of the same depth. It follows that neither $\mathfrak{L}_1 \leq \mathfrak{L}_2$ nor $\mathfrak{L}_2 \leq \mathfrak{L}_1$ holds. ☐

Theorem 3.26. *The following sets are definable.*

1. $\mathbb{L}$,
2. $\mathbb{D}$,
3. *the set of all frames from* $\mathbb{D}$ *(or* $\mathbb{L}$*) which have* n *points.*

Proof. From Lemma 3.21, Lemma 3.25 and Lemma 3.12. ☐

4. Linear $\mathbb{P}$–Frames

In this section we shall show that in fact every logic of a linear $\mathbb{P}$–frames is definable, from which we can deduce that any logic of linear $\mathbb{P}$–frames is invariant. To achieve this goal, we must first define a special class of frames:

Definition 4.1. *Let* $\mathfrak{F}$ *be a* $\mathbb{P}$*–frame. Call* x ***n–branching*** *if there exist* y_i*,* $i < n$*, pairwise different, such that* $x \lhd y_i$ *for all* $i < n$*, and if the generated frame of* y_i *contains* y_j*, then* $j = i$*.* $\mathbb{R}$ *denotes the class of* $\mathbb{P}$*–frames which are d–homogeneous and such that if a point* x *is 2–branching, then* x *is the root of* $\mathfrak{F}$*.*

Lemma 4.2. $\mathbb{R}$ *is definable.*

Proof. Let α be the following property: if $\mathfrak{G}$ and $\mathfrak{H}$ are nonlinear $\mathbb{C}^1$–cocovers of a linear frame, and $\mathfrak{G}, \mathfrak{H} \geq \mathfrak{F}$, then $\mathfrak{G} \cong \mathfrak{H}$. This is obviously an MSO–statement. Now, we shall show that for a d–homogeneous frame $\mathfrak{F}$, $\mathfrak{F}$ has α iff it is in $\mathbb{R}$. To that end, let $\mathfrak{F}$ be in $\mathbb{R}$ and let $\mathfrak{G}$ and $\mathfrak{H}$ be two frames which are nonlinear, but cocovers of linear frames. Assume further that $\mathfrak{G}, \mathfrak{H} \geq \mathfrak{F}$. If $\mathfrak{F}$ is linear, then these frames cannot exist. So, assume that $\mathfrak{F}$ is not linear. Then only the root is branching, and hence $\mathfrak{G}$ and $\mathfrak{H}$ have the same depth (which is the depth of $\mathfrak{F}$). Then, by d–homogeneity, $\mathfrak{G}$ and $\mathfrak{H}$ cover the same frame (up to isomorphism). Hence they are isomorphic, by Lemma 3.22. So, $\alpha(\mathfrak{F})$. Now assume that $\alpha(\mathfrak{F})$. Suppose that there are $\mathfrak{G}, \mathfrak{H} \geq \mathfrak{F}$, both $\mathbb{C}^1$–cocovers of a linear frame, and $\mathfrak{G} \ncong \mathfrak{H}$. Let $\mathfrak{L}$ be the linear cover of $\mathfrak{G}$, and $\mathfrak{M}$ the linear cover of $\mathfrak{H}$. Then, since $\mathfrak{F}$ is d–homogeneous, $\mathfrak{L} \geq \mathfrak{M}$ or $\mathfrak{M} \geq \mathfrak{L}$. Assume, without loss of generality, the first. However, by Lemma 3.22, the two are not isomorphic. It follows that $\mathfrak{L} > \mathfrak{M}$, and hence that $\mathfrak{L}$ has lesser depth than $\mathfrak{M}$. Now, let the depth of $\mathfrak{L}$ be k and the depth of $\mathfrak{M}$ be m. It follows that $\mathfrak{G}$ is of depth k as well and $\mathfrak{H}$ of depth m, and — finally — that $\mathfrak{F}$ is of depth $\geq m$ and $\mathfrak{F}$ has a branching point of depth $< m$. Thus, $\mathfrak{F}$ is not in $\mathbb{R}$. This completes the proof. ☐

Let $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ be a p–morphism. π is called **minimal**, if for any p–morphisms ρ, σ such that $\pi = \rho \circ \sigma$, either $\rho = \pi$ or $\sigma = \pi$. π is **2–collapsing**, if there are two points, x and y, $x \neq y$, such that: (a) $\pi(x) = \pi(y)$ and (b) for all $z, z' \notin \{x, y\}$: if $\pi(z) = \pi(z')$ then $z = z'$. The pair $\{x, y\}$

is called the **distinguished pair**. Obviously, a 2–collapsing p–morphism is minimal. If $\mathfrak{F}$ is d–homogeneous and $\mathfrak{G}$ a p–morphic image of $\mathfrak{F}$, then there exists a series of 2–collapsing p–morpisms mapping $\mathfrak{F}$ onto $\mathfrak{G}$. We distinguish two types of p–morphisms:

Type 1: The distinguished pair is $\{x, y\}$ and there is a z such that $z \lhd x, y$.

Type 2: The distinguished pair is $\{x, y\}$ and no z exists such that $z \lhd x, y$.

Obviously, if π is Type 1 and $z \lhd x, y$, z must be the root. It is easy to see the following:

Lemma 4.3. *Let $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ and $\rho : \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ be of Type 1. Then there exists $\mathfrak{G}'$ and $\rho' : \mathfrak{F} \twoheadrightarrow \mathfrak{G}'$ and $\pi' : \mathfrak{G}' \twoheadrightarrow \mathfrak{H}$ such that $\rho \circ \pi = \pi' \circ \rho'$.*

Recall that in general, for a 2–contracting p–morphism all we need to know to fix $\mathfrak{F}$ up to isomorphism is: (a) $\mathfrak{G}$, (b) the image point of the contracted pair, and (c) two sets A and B such that $A \cup B$ is the set C of predecessors of z. If C consists of one point only, we have a Type 1 p–morphism. For the sake of brevity, let us introduce the following notation.

Definition 4.4. *Let $\mathfrak{G}$ be a d–homogeneous frame, z a point and A and B sets such that $A \cup B$ is the set of predecessors of z. Then $W^{A,B}(\mathfrak{G})$ denotes the unique frame such that there is a 2–contracting p–morphism $\pi : W^{A,B}(\mathfrak{G}) \twoheadrightarrow \mathfrak{G}$ whose critical pair $\{x, y\}$ is mapped onto z, and A is the set of predecessors of x, B the set of predecessors of y. If A and B are clear from the context, they are omitted.*

Now let $\mathfrak{F}$ be a $\mathbb{R}$–frame and $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ be of Type 1. We claim that $\mathfrak{F}$ is uniquely determined (up to isomorphism) by the following data: (a) $\mathfrak{G}$, (b) the number k such that the for the critical pair $\{x, y\}$, both x and y are of depth k. (It is clear that x and y must be of the same depth.) Namely, we obviously have $\mathfrak{F} \restriction F - \{x, y\} \cong \mathfrak{G} \restriction G - \{\pi(x)\}$. We may actually consider these frames to be identical rather than isomorphic. Let $z := \pi(x)$. Let $u \neq x, y$. Then $u \lhd_F x$ iff $u \lhd_F x$ iff $u \lhd_G z$; $x \lhd_F u$ iff $y \lhd_F u$ iff $z \lhd_G u$. This establishes the claim. We denote $\mathfrak{F}$ by $\mathfrak{G}_k$. By the Lemma 4.3 it is established that $(\mathfrak{G}_m)_n \cong (\mathfrak{G}_n)_m$.

Let us take an $\mathbb{R}$–frame $\mathfrak{F}$. Then there exists a linear frame $\mathfrak{L}$ and a chain of 2–contracting p–morphisms mapping $\mathfrak{F}$ onto $\mathfrak{L}$. $\mathfrak{L}$ is uniquely determined, the chain is not. As remarked above, the members of the chain are either of Type 1 or of Type 2. Clearly, if $\mathfrak{F}$ is a cocover of $\mathfrak{L}$, the chain consists of a single 2–contracting p–morphism $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{L}$. This p–morphism is of Type 1.

Lemma 4.5. *Let $\mathfrak{L}$ be a linear frame and let $\mathfrak{F}$ be a $\mathbb{R}$–cocover of $\mathfrak{L}$. If $\mathfrak{F}$ is not linear, then $\mathfrak{F} \cong \mathfrak{L}_k$ for some k. Moreover, k is such that there exists a point of depth k in $\mathfrak{L}$ which is seen by the root.*

So, for every k such that the root sees a point of depth k, $\mathfrak{L}_k$ exists; and the frames of the form $\mathfrak{L}_k$ are the only $\mathbb{R}$–cocovers of $\mathfrak{L}$ which are not linear. It remains to establish a method such that, given a linear frame $\mathfrak{L}$ and a nonlinear $\mathbb{R}$–cocover $\mathfrak{F}$, we can establish the number k such that $\mathfrak{F} \cong \mathfrak{L}_k$. To that end, let us look at $\mathbb{R}$–frames which are in addition inherently 1–covered.

Definition 4.6. *A frame is called an $\mathbb{R}^n$**–frame** if it is an $\mathbb{R}$–frame and a $\mathbb{C}^n$–frame.*

Take the frame $\mathfrak{L}_k$ and an $\mathbb{R}^1$–cocover $\mathfrak{F}$. Then there exists a 2–contracting p–morphism $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{L}_k$. Suppose first that π is of Type 1. Then $\mathfrak{F} \cong \mathfrak{L}_{k,m}$ for some m. By Lemma 4.3 we must have $m = k$. So, $\mathfrak{F} \cong \mathfrak{L}_{k,k}$. It is easy to see that we have an infinite downgoing chain of $\mathbb{R}^1$–frames:

$$\mathfrak{L} > \mathfrak{L}_k > \mathfrak{L}_{k,k} > \mathfrak{L}_{k,k,k} > \dots$$

Now let us consider the case when π is of Type 2. If $\rho : \mathfrak{L}_k \twoheadrightarrow \mathfrak{L}$ has distinguished pair $\{x, y\}$, then $\{x, y\}$ is the set of predecessors of the point z which is doubled. If π is of Type 2, several choices appear for A and B (up to renaming): (A) $A = \{x, y\}$ and $B = \{x, y\}$, (B) $A = \{x\}$ and $B = \{y\}$. (C) $A = \{x\}$, $B = \{x, y\}$. However, since $\mathfrak{F}$ is a $\mathbb{R}$–frame, A and B can only contain one point. (Otherwise we have a point which is not the root and which is 2–branching.) Therefore, we are in situation (B). So, A and B are uniquely fixed, and we have $\mathfrak{F} \cong W(\mathfrak{L}_k)$.

Now assume that $\mathfrak{G}$ is an $\mathbb{R}$–cocover of $\mathfrak{F}$. If $\mathfrak{G} \cong \mathfrak{F}_m$ for some m, then — as is easy to verify — $\mathfrak{G}$ is not inherently 1–covered, since the maps $\mathfrak{G} \twoheadrightarrow \mathfrak{F} \twoheadrightarrow \mathfrak{L}_k$ 'commute'. This is asserted by the following lemma.

Lemma 4.7. *Let $\mathfrak{F}$ be an $\mathbb{R}$–frame and $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ and $\rho : \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ 2–contracting p–morphisms. Then if π is of Type 1 and ρ is of Type 2, there exists $\mathfrak{G}'$ and 2–contracting p–morphisms $\rho' : \mathfrak{F} \twoheadrightarrow \mathfrak{G}'$ and $\pi' : \mathfrak{G}' \twoheadrightarrow \mathfrak{H}$ such that $\pi' \circ \rho' = \rho \circ \pi$ and π' is of Type 1 and ρ' is of Type 2.*

Likewise, if we have $A = B$ of cardinality > 1, then we never get a $\mathbb{C}^1$–frame. Hence we conclude the following.

Lemma 4.8. *Let $\mathfrak{L}$ be linear and $k > 0$. $\mathfrak{L}_k$ has exactly two $\mathbb{R}^1$–cocovers: $\mathfrak{L}_{k,k}$ and $W(\mathfrak{L}_k)$. If $k = 0$, $\mathfrak{L}_k$ has exactly one $\mathbb{R}^1$–cocover.*

By induction, the following is established.

Lemma 4.9. *Let $k > 0$, $m > 0$. $W^m(\mathfrak{L}_k)$ has at most one $\mathbb{R}^1$–cocover. If $m = k$, it has no $\mathbb{R}^1$–cocover. If $m < k$, $W^{m+1}(\mathfrak{L}_k)$ is the unique $\mathbb{R}^1$–cocover of $W^m(\mathfrak{L}_k)$.*

Let us put this together. We are given a linear frame $\mathfrak{L}$. We want to establish its structure. First, we can determine its depth. Let it be d. We want to know, given two numbers $m, n \leq d$, whether the point of depth m sees a point of depth n. To that end we establish first the generated subframe of depth m. (This is easily construced; just go $d - m$ steps up in $\langle \mathbb{I}, \leq \rangle$.) For simplicity, we assume $m = d$. Now, take an $\mathbb{R}^1$–cocover $\mathfrak{M}$ of $\mathfrak{L}$. If it has exactly one $\mathbb{R}^1$–cocover, then $\mathfrak{M} \cong \mathfrak{L}_0$, and the root sees a point of depth 0. Otherwise, $\mathfrak{M}$ has two $\mathbb{R}^1$–cocovers, $\mathfrak{N}_1$ and $\mathfrak{N}_2$. One of them, say $\mathfrak{N}_1$, has an infinite downgoing chain of $\mathbb{R}^1$–frames below it. Now take the number p of $\mathbb{R}^1$–frames strictly below $\mathfrak{N}_2$. Then $\mathfrak{M} \cong \mathfrak{L}_{p+1}$. In this way we can find out all the numbers n such that the root sees a point of depth n. This is summarized in the next theorem.

Theorem 4.10. *Let Θ be the logic of a linear frame. Then Θ is definable.*

Corollary 4.11. *Let Θ be the logic of a given set of linear frames. Then Θ is invariant. In particular, **K.alt**$_1$ and every extension of **G.3** is invariant.*

Let us cash out a little bit further on the results established so far. Notice that the logics $\mathsf{Th}\,\mathfrak{L}_{k,k,\ldots}$ are all definable. This opens the way for the following result.

Definition 4.12. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame and x a point of depth d. We say that x is **directly** n–**branching** if x has exactly n successors of depth $d - 1$.*

Theorem 4.13. *The class of d–homogeneous frames of depth $\geq \delta$ which have an at least (at most, exactly) directly n–branching point of depth d is definable.*

Proof. Let $\mathfrak{F} \leq \mathfrak{L}_{k^{n-1}}$ for some linear frame $\mathfrak{L}$ of depth $\leq \delta$. Then it is immediately verified that $\mathfrak{F}$ contains a point of depth $\leq \delta$ which is immediately $\geq n$–branching. Conversely, suppose that $\mathfrak{F}$ contains an immediately at least n–branching point x of depth $d \leq \delta$. Take $\mathfrak{G} :=\uparrow x$. For all $k < d - 1$, contract all points of depth k onto a single point. The resulting frame is isomorphic to a frame $\mathfrak{L}_{k^p}$ for some $\mathfrak{L}$ of dept d and some $p \geq n$. This frame is contractible to a frame $\mathfrak{L}_{k^n}$. ☐

Lemma 4.14. *Let $\mathfrak{F}$ be a d–homogeneous frame of width $\geq n$. Then there exists a contraction image $\mathfrak{G}$ which has width exactly n. There is a unique largest frame with this property.*

Theorem 4.15. *The class of d–homogeneous frames of depth $\leq \delta$ which have at least (at most, exactly) n points of depth d is definable.*

Proof. The proof is by induction on d. Suppose the claim is true of all $k < d$. Then take the frame $\mathfrak{G} \geq \mathfrak{F}$ of largest size such that it has exactly one point of depth k for all $k < d$. Since the number of points of depth $k < d$ can be established by induction hypothesis, we can also construct $\mathfrak{G}$, since it is uniquely defined by its description, by Lemma 4.14. It is the result of reducing $\mathfrak{F}$ to a linear frame at depth $< d$. This frame exists, again by Lemma 4.14. (The reader is asked to note the special case $d = 0$, where is fact nothing needs to be proved, and $\mathfrak{G} \cong \mathfrak{F}$.) Let $\mathbb{H}_{\delta,d}$ the set of all $\mathbb{R}$–frames of depth $\leq \delta$, width 2, which have exactly one point of depth k for all $k < d$. This set is finite, and it is definable. Now, we claim that for all d–homogeneous $\mathfrak{F}$: $\mathfrak{F}$ has only one point of depth d iff $\mathfrak{F} \not\leq \mathfrak{H}$ for all $\mathfrak{H} \in \mathbb{H}_{\delta,d}$. For let $\mathfrak{F} \leq \mathfrak{H}$ for some $\mathfrak{H} \in \mathbb{H}_{\delta,d}$. Then clearly $\mathfrak{F}$ must have two points of depth d. Now assume that $\mathfrak{F}$ has two points of depth d. We contract $\mathfrak{F}$ to a frame of width 2, and of width 1 at all depths $< d$ and call the result $\mathfrak{F}^1$. Let x and y be the points of depth d. Let z be the point of least depth such that $\mathfrak{F}^1 \uparrow z$ contains x and y. Then $\mathfrak{F}^1$ is an $\mathbb{R}$–frame; it is of width 2 and has width 1 at all depths $< d$. Further, $\mathfrak{F}^1 \uparrow z$ has depth $\leq \delta$. So, it is in $\mathbb{H}_{\delta,d}$. This shows that $\mathfrak{F} \leq \mathfrak{H}$ for some $\mathfrak{H} \in \mathbb{H}_{\delta,d}$, as promised. Finally, we need to find the number of points of depth d. In order to do this, let us return to $\mathfrak{F}$ and $\mathfrak{G}$ as constructed above. Let $\mathfrak{H}$ be the largest frame such that $\mathfrak{H} \geq \mathfrak{G}$ and such that $\mathfrak{H}$ has exactly one point at depth d. This frame is uniquely defined. Let m be the size of the longest chain of frames between $\mathfrak{G}$ and $\mathfrak{H}$ (including $\mathfrak{G}$ and $\mathfrak{H}$). Then m is the number of points at depth d. For the contraction of all points of depth d to a single point can be split into a series of $d-1$ 2–contracting p–morphisms, giving rise to a chain of length d. On the other hand, in between $\mathfrak{G}$ and $\mathfrak{H}$ no other frames exist, that is to say, $\mathfrak{H}$ has the same number of points at all depths $> d$ as $\mathfrak{G}$. This finally concludes the proof. ☐

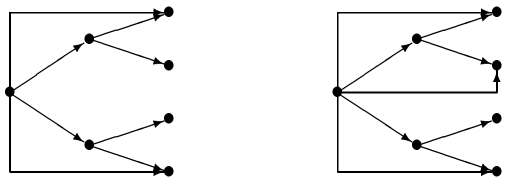
This allows us to define another set of very important frames, namely homogeneously branching trees.

Definition 4.16. *A $\mathbb{P}$–frame $\mathfrak{F}$ is a* ***tree*** *if for all x and y we either have $\mathfrak{F} \uparrow x \subseteq \mathfrak{F} \uparrow y$ or $\mathfrak{F} \uparrow y \subseteq \mathfrak{F} \uparrow x$ or $\mathfrak{F} \uparrow x \cap \mathfrak{F} \uparrow y = \varnothing$. $\mathfrak{F}$ is* ***homogeneously n–branching*** *if every point of depth > 0 is directly n–branching.*

Theorem 4.17. *Let $\mathbb{T}$ be the class of a d–homogeneous, homogeneously n–branching tree of depth δ. Then $\mathsf{Th}\,\mathbb{T}$ is invariant.*

Proof. $\mathfrak{F} \in \mathbb{T}$ iff it is of depth δ, is d–homogeneous, n–branching at each depth and has exactly $n^{\delta-h}$ points of depth h. This set is definable by Theorems 3.26, 4.13 and 4.15. ☐

Unless the depth profile consists only of the pairs $\langle n, n-1\rangle$, this does not fix the trees up to isomoprhism. The following trees are d–homogeneous, homogeneously 2–branching, and have the same depth profile.



It requires considerable effort to establish the definability of each individual frame of this kind. We will not go into the details here and concentrate on the linear frames instead.

However, if the depth profile is minimal, then this fixes the tree up to isomorphism. The logics $\mathbf{K.alt}_n.\mathbf{D}$ are characterhized by the set of homogeneously n–branching trees. Hence we obtain

Corollary 4.18. *The logics $\boldsymbol{K.alt}_1.\boldsymbol{D}$ are invariant.*

5. Logics of Depth 1

The results of the previous section can be pushed up remarkably by the following technique. Let P be a set of logics, and let Θ be a logic. Then a subset $Q \subseteq P$ is called a P–**subreduction** of Θ if $\bigsqcap Q \leq \Theta$. A set H of logics is P–**stable** if it is a set of logics having the same set of P–subreductions. Θ is called P–stable if $\{\Theta\}$ is P–stable. Then the following is immediate from Lemma 2.5.

Lemma 5.1. *Assume that P is a set of definable logics. If Θ is P–stable, it is invariant.*

The content of the concept of P–stability is the following. Although we have immediate hopes only to define logics of cycle–free frames, we can extend this to other logics as well. Take a frame $\mathfrak{F}$, not cycle–free. Then, by unravelling, there is a sequence $\mathfrak{P}_n$ of cycle–free frames such that $\bigsqcap \mathsf{Th}\,\mathfrak{P}_n \leq \mathsf{Th}\,\mathfrak{F}$. Therefore, $\{\mathsf{Th}\,\mathfrak{P}_n : n \in \omega\}$ forms a $\mathbb{P}$–subreduction of $\mathsf{Th}\,\mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{G}$ have different unravellings, then no automorphism can map $\mathfrak{F}$ onto $\mathfrak{G}$. In particular, if no logic has the same unravellings as $\mathfrak{F}$, $\mathfrak{F}$ will remain invariant.

We shall precisify the notion of unravelling as follows. Let $\mathfrak{F}$ be a Kripke–frame, and $x \in F$. Then denote by $T^n(x, \mathfrak{F})$ the set of points that can be reached in at most n steps from x. This set is called the n–**transit** of x in $\mathfrak{F}$ (see [2]). The subframe $\mathfrak{T}^n(x, \mathfrak{F}) := \mathfrak{F} \upharpoonright T^n(x, \mathfrak{F})$ is likewise called the n–**transit** of x in $\mathfrak{F}$. We omit $\mathfrak{F}$ whenever it is clear which frame is meant. The following is easily shown by induction on n.

Lemma 5.2. *Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are frames, $x \in F$ and $y \in G$ points and $\mathfrak{T}^n(x, \mathfrak{F}) \cong \mathfrak{T}^n(y, \mathfrak{G})$. Then for every formula φ of modal degree $\leq n$:*

$$\langle \mathfrak{F}, x\rangle \models \varphi \quad \Leftrightarrow \quad \langle \mathfrak{G}, y\rangle \models \varphi$$

Definition 5.3. *An* ***unravelling of*** *$\mathfrak{F}$ is an infinite series $\underline{G} = \langle \mathfrak{G}_n : n \in \omega\rangle$ of frames such that the following holds.*

1. *For all $n \in \omega$: $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$.*
2. *For all $n \in \omega$: $\mathfrak{G}_n \not\leq \mathfrak{F}$.*
3. *For each $m, n \in \omega$ and $x \in G_n$ there exists a p such that for all $q \geq p$:*
$$\mathfrak{T}^m(x, \mathfrak{G}_q) = \mathfrak{T}^m(x, \mathfrak{G}_p)\ .$$
4. $\lim \underline{G} \leq \mathfrak{F}$.
5. *There exists a $k \in \omega$ such that for all $q \in \omega$ and $x \in G_{q+k} - G_k$ there exists a $y \in G_q$ with*
$$\mathfrak{G}_{q+k} \upharpoonright x \cong \mathfrak{G}_q \upharpoonright y\ .$$

The last condition is called the ***cyclicity condition****. $\underline{G}$ is called* ***finite*** *if all $\mathfrak{G}_k$ are finite.*

Here, $\lim \underline{G}$ is the frame $\langle \bigcup_n G_n, \bigcup_n \lhd_n\rangle$. Obviously, unravellings are some kind of subreductions. However, the existence of some subreduction is not enough. What we must establish for stability (and invariance) is which frames a given sequence of frames subreduces. This is what unravellings allow us to do.

The following is an easy consequence of cyclicity.

Lemma 5.4. *Let $\underline{G}$ be an unravelling. Suppose that $\langle \mathfrak{G}_q, x\rangle \models \varphi$. Then there exists a $p \leq q$ and a $y \in G_k$ such that $\langle \mathfrak{G}_p, y\rangle \models \varphi$.*

Proof. Suppose $\langle \mathfrak{G}_q, x\rangle \models \varphi$. If $x \in G_k$ we are done. Otherwise, we have $q \geq k+1$ and there exists a $x' \in G_{q-k}$ such that $\mathfrak{G}_{q-k} \upharpoonright x' \cong \mathfrak{G}_q \upharpoonright x$. It follows that for this x', $\langle \mathfrak{G}_{q-k}, x'\rangle \models \varphi$. Now reason with x' instead of x and $q - k$ instead of q. □

Lemma 5.5. *Let $\underline{G}$ be an unravelling of $\mathfrak{F}$. Then, given $x \in G_n$ and some formula φ we either have $\langle \mathfrak{G}_q, x\rangle \models \varphi$ for finitely many q, or for almost all q.*

Proof. Let φ be given. Suppose φ has modal depth m. Then there exists an p such that for all $q \geq p$: $\mathfrak{T}^m(x, \mathfrak{G}_q) = \mathfrak{T}^m(x, \mathfrak{G}_p)$. Whence by Lemma 5.2

$$\langle \mathfrak{G}_q, x\rangle \models \varphi \quad \Leftrightarrow \quad \langle \mathfrak{G}_p, x\rangle \models \varphi$$

□

Lemma 5.6. *Let $\underline{G}$ be an unravelling of $\mathfrak{F}$ and $x \in \mathfrak{G}_n$. Then $\langle \lim \mathfrak{G}_n, x\rangle \models \varphi$ iff $\langle \mathfrak{G}_q, x\rangle \models \varphi$ for almost all q.*

Proof. Notice that $\mathfrak{T}^m(x, \lim \underline{G}) = \mathfrak{T}^m(x, \mathfrak{G}_q)$ for almost all q. □

Consider the formulae $\varphi_n := p_y \wedge \Box^n \delta(\mathfrak{G})$, where $\delta(\mathfrak{G})$ is the diagram of $\mathfrak{G}$. This is defined as follows.

$$\begin{array}{rll} \delta(\mathfrak{G}) := & & \bigvee\langle p_y : y \in G\rangle \\ & \wedge & \bigwedge\langle p_x \to \neg p_y : x \neq y\rangle \\ & \wedge & \bigwedge\langle p_x \to \Diamond p_y : x \lhd y\rangle \\ & \wedge & \bigwedge\langle p_x \to \neg\Diamond p_y : x \not\lhd y\rangle \end{array}$$

Definition 5.7. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames, $x \in F$, and $h : F \to G$ a map. h is n–**localic with respect to** x, if the following holds.*

1. *If $y \lhd^F z$ for $y, z \in \mathfrak{T}^n(\mathfrak{F}, x)$ then $h(y) \lhd^G h(z)$.*
2. *If $y \in \mathfrak{T}^{n-1}(\mathfrak{F}, x)$ and $h(y) \lhd^G u$ then there exists a $z \in F$ such that $y \lhd^F z$ and $h(z) = u$.*

Lemma 5.8. *$p_y \wedge \Box^n\delta(\mathfrak{G})$ can be satisfied at x in $\mathfrak{F}$ iff there exists an n–localic map from $\mathfrak{F} \uparrow y$ onto $\mathfrak{G} \uparrow x$.*

The following can be found in Kracht [2].

Theorem 5.9. *Let $\mathfrak{G}$ be finite frame. $\Delta := \{p_y \wedge \Box^n\delta(\mathfrak{G}) : n \in \omega\}$ is satisfiable in a frame $\mathfrak{F}$ iff $\mathfrak{G}$ is a generated subframe of some contraction of an ultraproduct of $\mathfrak{F}$ iff $\mathfrak{G} \geq \mathfrak{F}$. If $\mathfrak{F}$ is finite, then $\mathfrak{G}$ is the contraction image of some generated subframe of $\mathfrak{F}$.*

Lemma 5.10. *Let $\underline{G}$ be a finite unravelling of $\mathfrak{F}$ and $\mathfrak{K}$ a finite frame. Then there exists a point x and a valuation β such that $\langle \lim \underline{G}, \beta, x\rangle \models \{p_y \wedge \Box^n\delta(\mathfrak{K}) : n \in \omega\}$ iff for all n, $p_y \wedge \Box^n\delta(\mathfrak{K})$ is satisfiable in almost all $\mathfrak{G}_m$.*

Proof. ($\Rightarrow$) Suppose that $\langle \lim \underline{G}, \beta, x\rangle \models \{p_y \wedge \Box^m\delta(\mathfrak{K}) : m \in \omega\}$ for some x and some β. Let m be given. Then there exists a q such that $\mathfrak{T}^{m+1}(\mathfrak{G}_p, x) \cong \mathfrak{T}^{m+1}(\lim \underline{G}, x)$. Hence, for the restriction β_q of β to $\mathfrak{G}_q$ we have $\langle \mathfrak{G}_q, \beta_q, x\rangle \models p_y \wedge \Box^m\delta(\mathfrak{K})$. So, for each m, $p_y \wedge \Box^m\delta(\mathfrak{K})$ is satisfiable in at least some member of the unravelling. ($\Leftarrow$) Suppose that $p_y \wedge \Box^n\delta(\mathfrak{K})$ is satisfiable in almost all $\mathfrak{G}_m$. By Lemma 5.4, we may assume that $p_y \wedge \Box^n\delta(\mathfrak{K})$ is satisfiable in some $\mathfrak{G}_p$ at a point in G_k. Since G_k is finite, there must be a point $x \in G_k$ such that x satisfies for each m the formula $p_y \wedge \Box^m\delta(\mathfrak{K})$ in $\mathfrak{G}_{p_m}$ for some p_m (under some valuation β_m). It is not necessarily the case that $\beta_{m+1}(p_x) \cap G_{p_m} = \beta_m(p_x)$ for all $x \in K$. However, by choosing an appropriate subsequence we get (for infinitely many and so) for each m a model

$$\langle \mathfrak{G}_{p_m}, \beta_m, x\rangle \models p_y \wedge \Box^m\delta(\mathfrak{K})$$

and $\beta_n(p_x) \cap G_m = \beta_m(p_x)$ for all $x \in K$ and all m, n such that $n \geq m$. Put $\beta(p_x) := \bigcup\langle \beta_m(p_x) : m \in \omega\rangle$. This is a valuation into $\lim \underline{G}$. Choosing this valuation we now have

$$\langle \lim \underline{G}, \beta, x\rangle \models \{p_y \wedge \Box^m\delta(\mathfrak{K}) : m \in \omega\}\ .$$

∎

Theorem 5.11. *Let $\underline{G} = \langle \mathfrak{G}_n : n \in \omega\rangle$ be a finite unravelling of $\mathfrak{F}$. Then the following holds for any finite rooted frame $\mathfrak{H}$: either (a) $\mathfrak{H} \geq \mathfrak{G}_n$ for some n, or (b) $\mathfrak{H} \geq \lim \underline{G}$, or (c) $\mathfrak{H}$ is not subreduced by $\underline{G}$.*

Proof. Consider $\Delta := \{p_y \wedge \Box^m\delta(\mathfrak{H}) : m \in \omega\}$, where y is the root of $\mathfrak{H}$. (Case 1) Δ is simultaneously satisfiable in some $\mathfrak{G}_n$. Then, since $\mathfrak{G}_n$ is finite, some generated subframe of $\mathfrak{G}_n$ can be mapped onto $\mathfrak{H}$, and $\mathfrak{H} \leq \mathfrak{G}_n$. (Case 2) Δ is not simultaneously satisfiable in some $\mathfrak{G}_n$, but every member of Δ is satisfiable somewhere. It follows by Lemma 5.10 that every member is almost always satisfiable, and so there exists a point $x \in \lim \underline{G}$ such that Δ is satisfiable at x in $\lim \underline{G}$. So, there exists a series $\pi^m : \lim \underline{G} \twoheadrightarrow \mathfrak{H}$ of maps such that π^m is m–localic with respect to x for each m. Now, for each n there exists an infinite subset $N_m \subseteq \omega$ such that $\pi^p \restriction T^m(\lim \underline{G}, x) = \pi^q \restriction T^m(\lim \underline{G}, x)$ for all $p, q \in N_m$. We may choose N_m inductively such that $N_m \supseteq N_{m+1}$. Define $\rho : \lim \underline{G} \twoheadrightarrow \mathfrak{H}$ as follows. If $y \in T^m(\lim \underline{G}, x)$ then $\rho(y) := \pi^p(y)$, where $p \in N_m$, arbitrary. By choice of the sets N_m, this does not depend on the number p. One easily shows that ρ is m–localic with respect to x for each m, which is to say that it is a p–morphism. (Case 3) Almost all members of Δ are nowhere satisfiable in any of $\mathfrak{G}_n$. Then $\mathfrak{H}$ is not subreduced by $\underline{G}$, by the remarks above. ∎

Corollary 5.12. *Let $\underline{G}$ be a finite unravelling. Suppose that all $\mathfrak{G}_m$ are invariant. Then* $\lim G$ *is invariant.*

Proof. $\lim \underline{G}$ is an intersection of logics with finite model property, and therefore has the finite model property as well. From Theorem 5.11 it follows that $\mathsf{Th}\lim \underline{G} \geq \bigcap \langle \mathsf{Th}\,\mathfrak{G}_m : m \in \omega\rangle$. The latter is invariant, by Lemma 2.5. Now, $\mathsf{Th}\lim \underline{G}$ is the intersection of the logics of all finite frames which are above the intersection of all $\mathsf{Th}\,\mathfrak{G}_m$, but not above any $\mathfrak{G}_m$. This is again invariant. ☐

We shall apply this technique to cyclic frames.

Definition 5.13. *A Kripke–frame $\mathfrak{F}$ is called* ***cyclic*** *if it is not the singleton irreflexive point and $\mathfrak{F}$ possesses at most two nonisomorphic generated subframes: the empty frame, and $\mathfrak{F}$ itself.*

The following theorem shows why the name cyclic is appropriate.

Lemma 5.14. *Let $\mathfrak{F}$ be a finite cyclic Kripke–frame. Then $F = \{c_i : i < n\}$ and $\lhd \supseteq \{\langle c_i, c_j\rangle : j \equiv i + 1 \pmod n\}$.*

Let $\mathfrak{F}$ be a finite, cyclic Kripke–frame. Then put $\mathfrak{U}_\omega(\mathfrak{F}) := \langle \omega, \blacktriangleleft\rangle$, where $p \blacktriangleleft q$ iff (a) $q = p+1$ or (b) $p = \gamma n + j$, $q = \gamma n + k$ for some $\gamma \in \omega$, $j < k < n$ and $c_j \lhd c_k$ or (c) $p = \gamma n + j$, and $q = (\gamma + 1)n + k$ for some $\gamma \in \omega$, $k \leq j < n$ and $c_j \blacktriangleleft c_k$. $\mathfrak{U}_p(\mathfrak{F}) := \mathfrak{U}_\omega(\mathfrak{F}) \restriction \{0, 1, \dots , p-1\}$.

Lemma 5.15. *$\{\mathfrak{U}_p(\mathfrak{F}) : p \in \omega\}$ is an unravelling and an $\mathbb{L}$–subreduction of $\mathfrak{F}$.*

Proof. The proof is a longish verification of the details. We shall only some of the work. Clearly, $\mathfrak{U}_p(\mathfrak{F})$ is linear for every $p \in \omega$, so we need to show only that we have an unravelling. Condition 1 is obviously satisfied. Condition 2 is also easy. The unravelling frames are cycle free. For Condition 3 $p \geq m + n$ is clearly sufficient. For Condition 4 one only needs to show that there exists a p–morphism from $\mathfrak{U}_\omega(\mathfrak{F})$ onto $\mathfrak{F}$. But this is easy. Finally, Condition 5. Put $k := n$, where $k := n$ and n is the size of the cycle. Consider the point j of $\mathfrak{U}_{q+n}(\mathfrak{F})$ Suppose that $j \geq n$, then $\mathfrak{U}_{q+n}(\mathfrak{F}) \restriction j \cong \mathfrak{U}_q(\mathfrak{F}) \restriction j - n$, as is easily verified. (The isomorphism is given by $i \mapsto i - n$.) ☐

We shall start with the logics of the frames $\mathfrak{Loop}_n$. The method above yields uniformly the frame $\langle \omega, \blacktriangleleft\rangle$, where $i \blacktriangleleft j$ iff $j = i + 1$. So, using $\mathfrak{U}_\omega$ we cannot discriminate between any of these loops. Fortunately, we can revise our definition as follows. Let $\mathfrak{U}^1_\omega(\mathfrak{F})$ differ from $\mathfrak{F}$ in that for each $\langle i, j\rangle \in\blacktriangleleft$, the arc $\langle i, i+j+n\rangle$ is added. Let $\mathfrak{U}^1_q(\mathfrak{F})$ be the subframe of the first q points. It is shown as above that $\{\mathfrak{U}^1_q(\mathfrak{F}) : q \in \omega\}$ is an $\mathbb{L}$–subreduction of $\mathfrak{F}$ if $\mathfrak{F}$ is cyclic. Now consider a p–morphism $\pi : \mathfrak{U}^1_\omega(\mathfrak{Loop}_n) \twoheadrightarrow \mathfrak{Loop}_m$. Take a point j. It has two successors, $j + 1$ and $j + 1 + n$. However, $\pi(j)$ has only one successor. So, $\pi(j + 1 + n) = \pi(j + 1)$, for all j. So we conclude that $m \leq n$. (Since π factors through the natural projection, $\rho_n : \mathfrak{U}^1_\omega(\mathfrak{Loop}_n) \twoheadrightarrow \mathfrak{Loop}_n$ we even have $m|n$.) So, let $Q_n := \{\mathfrak{U}^1_q(\mathfrak{Loop}_n) : q \in \omega\}$. Then Q_n is not a subreduction of any $\mathfrak{Loop}_m$ where $m > n$. (In fact, this needs some careful argumentation. First, Q_n is a sequence of $\mathbf{K.alt}_2$–logics. Next, by Jónssons Theorem, $\mathsf{Th}\,\mathfrak{Loop}_m \geq \bigcap_q \mathsf{Th}\,\mathfrak{U}^1_q(\mathfrak{Loop}_n)$ implies that the algebra of $\mathfrak{Loop}_m$ is a generated subframe of some p–morphic image of an ultraproduct of these frames. We may commute the operators H and S. Now, the ultraproduct has the property that every generated component looks like $\mathfrak{U}^1_\omega(\mathfrak{Loop}_n)$ or some $\mathfrak{U}^1_q(\mathfrak{Loop}_n)$. Any generated subframe has the same property. Hence, we may reason the way we did.) Thus, we have shown that all loops have different subreduction sequences. A last detail is missing. Let us note that if a frame has the subreduction sequence $\{\mathfrak{Ch}_q : q \in \omega\}$ iff it is an $\mathbf{K.alt}_1$–frame, we see that each cyclic $\mathbf{K.alt}_1$–frame is uniquely characterized by its $\mathbb{L}$–subreduction sequence.

Theorem 5.16. *Every extension of $\boldsymbol{K.alt}_1$ is invariant.*

Proof. We may restrict our attention to $\bigsqcap$–irreducible logics. These are the logics of the chains, or of the frames $\mathfrak{Loop}_{p,q} := \langle \{0, 1, \dots , p + q\}, \lhd\rangle$, where $i \blacktriangleleft j$ iff $j = i + 1$ or $i = p + q$ and $j = p$. In our previous notation, $\mathfrak{Loop}_n = \mathfrak{Loop}_{0,n}$. We have established that $\mathfrak{Loop}_{0,n}$ is invariant for all n. Now observe that $\mathfrak{Loop}_{p,n}$ is characterized by the following: (A) its logic is $\sqcap$–irreducible, (B) the largest loop $\geq \mathfrak{Loop}_{p,n}$ is $\mathfrak{Loop}_{0,n}$, (C) there are $p+1$ many irreducible logics between $\mathfrak{Loop}_{p,n}$

and $\mathfrak{Loop}_{0,p}$, namely all logics of the form $\mathfrak{Loop}_{j,n}$. Now it is not hard to see that also $\mathfrak{Loop}_{p,n}$ is invariant for all p and n. □

This does not yet establish that all logics of depth 1 are invariant. But this will follow from the results of the next section, which are even far more general.

6. Linear Logics

Finally, let us extend the notion of linearity to all finite Kripke–frames in the following way.

Definition 6.1. *Let $\mathfrak{F}$ be a finite Kripke–frame and x and y points of $\mathfrak{F}$. x and y are called **g–equivalent** if $\mathfrak{F} \uparrow x = \mathfrak{F} \uparrow y$. x is of **depth 1** if every $y \in \mathfrak{F} \uparrow x$ is g–equivalent with x. x is of **depth $n + 1$** iff for all $y \in \mathfrak{F} \uparrow x$: either y is of depth n or y is g–equivalent to x. $\mathfrak{F}$ is of **depth n** if it has a point of depth n but no point of depth $n + 1$.*

Definition 6.2. *Let $\mathfrak{F}$ be a finite Kripke–frame. $\mathfrak{F}$ is **linear** if for all m and n such that $m \leq n$: if x is of depth n and y of depth m, then $y \in \mathfrak{F} \uparrow x$.*

Now we are going to prove that every logic of linear frames is invariant. The proof will be done by induction in the number of cycles.

Definition 6.3. *A **cycle** is a maximal set of g–equivalent points which is not identical to a set containing a single irreflexive point. The number of cycles of a given frame $\mathfrak{F}$ is denoted by $cyc(\mathfrak{F})$.*

Consider the frame $\mathfrak{F}$. Suppose, it is generated by x, and that x is contained in a cycle C. We assume that $C = \{c_i : i < n\}$, where $c_i \lhd c_j$ for $j \equiv i + 1 \pmod n$. Without loss of generality, $x = c_0$. We construct the following frames.

$$\begin{array}{rcl} U(\mathfrak{F}) & := & \langle U(F), \lhd^U \rangle \\ V(\mathfrak{F}) & := & \langle U(F), \lhd^V \rangle \end{array}$$

Here,

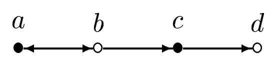
$$U(F) := n \times \{0\} \cup C \times \omega \times \{1\} \cup (F - C) \times 2$$

The relation is defined as follows.

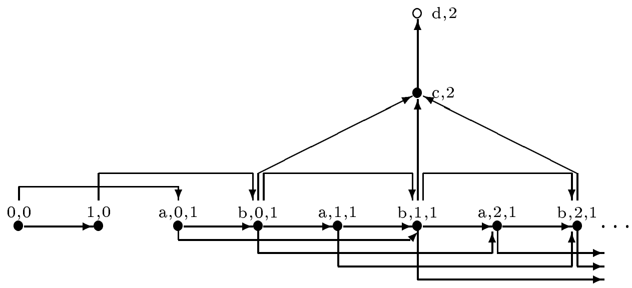
$$\lhd^U := \left\{ \begin{array}{ll} & \{\langle \langle i, 0 \rangle, \langle i + 1 \rangle \rangle : i + 1 < n\} \\ \cup & \{\langle \langle i, 0 \rangle, \langle c_i, 0, 1 \rangle \rangle : i < n\} \\ \cup & \{\langle \langle c_i, j, 1 \rangle, \langle c_k, j, 1 \rangle \rangle : i < k, c_i \lhd c_k\} \\ \cup & \{\langle \langle c_i, j, 1 \rangle, \langle c_k, j + 1, 1 \rangle \rangle : c_i \lhd c_k\} \\ \cup & \{\langle c_i, j, 1 \rangle, \langle x, 2 \rangle \rangle : c_i \lhd x\} \\ \cup & \{\langle \langle x, 2 \rangle, \langle y, 2 \rangle \rangle : x \lhd y\} \end{array} \right.$$

$$\lhd^V := \lhd^U \cup \{\langle \langle i, 0 \rangle, \langle c_i, j, 1 \rangle \rangle : i < n, j \in \omega\}$$

To give an example, consider the following frame. (Here, • denotes an irreflexive and ∘ a reflexive point.)



Then $U(\mathfrak{F})$ has the following form (where $c_0 = a$, $c_1 = b$).



The frames $U^p(\mathfrak{F})$ and $V^p(\mathfrak{F})$ are obtained from the frames $U(\mathfrak{F})$ and $V(\mathfrak{F})$ by restricting the set of worlds to the set $n \times \{0\} \cup C \times p \times \{1\} \cup (F - C) \times \{2\}$. Further, we let $U^{-k}(\mathfrak{F})$ (and $V^{-k}(\mathfrak{F})$) denote the subframe obtained by removing the points $\langle i, 0\rangle$ for all $i < k$. It is clear from the results of the previous section that $U^p(\mathfrak{F})$ subreduces $U(\mathfrak{F})$ and $V^p(\mathfrak{F})$, since they are unravellings. Moreover, for a finite rooted $\mathfrak{G}$ the following holds by Theorem 5.11. $\mathfrak{G}$ is subreduced by $\{U^p(\mathfrak{F}) : p \in \omega\}$ if (1) $\mathfrak{G} \geq U^p(\mathfrak{F})$ for no $p \in \omega$, and (2) $\mathfrak{G} \geq U(\mathfrak{F})$.

Notice that $cyc(U^p(\mathfrak{F})) = cyc(\mathfrak{F}) - 1$. We shall investigate the logics of finite frames that are subreduced by $U^p(\mathfrak{F})$ and $V^p(\mathfrak{F})$.

Definition 6.4. *Let $\mathbb{S}(\mathfrak{F})$ be the set of finite $\mathbb{I}$–frames subreduced both by $\{U^p(\mathfrak{F}) : p \in \omega\}$ and $\{V^p(\mathfrak{F}) : p \in \omega\}$.*

$\mathbb{S}(\mathfrak{F})$ is an upper subset of $\langle \mathbb{I}, \leq\rangle$.

Lemma 6.5. *Suppose that $\mathfrak{G}$ is generated by a cycle. Then $\mathfrak{G}$ has infinite dimension in $\mathbb{S}(\mathfrak{F})$.*

Proof. Consider the following frames. $T^k(\mathfrak{F}) := \langle T^k(F), \lhd^k\rangle$, defined by

$$T^k(\mathfrak{F}) := C \times k \cup (F - C) \times \{k\}$$

$$\lhd^k := \left\{ \begin{array}{ll} & \{\langle\langle c_i, j\rangle, \langle c_{i'}, j+1\rangle\rangle : c_i \lhd c_{i'}, j+1 < k\} \\ \cup & \{\langle\langle c_i, k-1\rangle, \langle c_{i'}, 0\rangle\rangle : c_i \lhd c_{i'}\} \\ \cup & \{\langle\langle c_i, j\rangle, \langle c_{i'}, j\rangle\rangle : c_i \lhd c_{i'}, i < i'\} \end{array} \right.$$

Take the generated subframe $U^{-n}(\mathfrak{F})$ of $U(\mathfrak{F})$ consists of all points with second component 1 or 2. This frame is isomorphic to the generated subframe $V^{-k}(\mathfrak{F})$ of $V(\mathfrak{F})$. This frame can be mapped p–morphically onto $T^k(\mathfrak{F})$. (Namely, put $\pi(\langle x, 2\rangle) := \langle x, k\rangle$, and $\pi(\langle c_i, \lambda k + q, 1\rangle) := \langle c_i, q\rangle$, $q < k$.) Hence, $T^k(\mathfrak{F})$ is subreduced by both sets, and is member of $\mathbb{S}(\mathfrak{F})$. Now, $T^k(\mathfrak{F}) \leq T^{k'}(\mathfrak{F})$ if k is a multiple of k'. (Again, there is a p–morphism from the latter onto the former.) This shows the claim. ☐

Lemma 6.6. *Suppose that $\mathfrak{G}$ is a minimal member of $\mathbb{S}(\mathfrak{F})$ and $cyc(\mathfrak{G}) = cyc(\mathfrak{F})$. Then $\mathfrak{G}$ is a p–morphic image of $U(\mathfrak{F})$ (and $V(\mathfrak{F})$), and $\mathfrak{F}$ a generated subframe of $\mathfrak{G}$.*

Proof. Now, first of all, $\mathfrak{G}$ is not generated by a cycle, by the previous lemma. By the condition that $cyc(\mathfrak{G}) = cyc(\mathfrak{F})$ we deduce that $\mathfrak{G} \geq U(\mathfrak{F})$ as well as $\mathfrak{G} \geq V(\mathfrak{F})$. Moreover, we must have a p–morphism $\pi : U^{-k}(\mathfrak{F}) \twoheadrightarrow \mathfrak{G}$ and $\rho : V^{-k}(\mathfrak{F}) \twoheadrightarrow \mathfrak{G}$ for some $k < n$, by these assumptions. Clearly, by minimality of $\mathfrak{G}$, $k = 0$. Consider now the point $\langle k, 0\rangle$. In the first frame it sees at most two points, in the latter infinitely many. It follows that $\rho(\langle c_k, \lambda, 1\rangle) = \rho(\langle c_k, \mu, 1\rangle)$ for all $\lambda, \mu \in \omega$ and for all $k < n$. Hence, there is a cycle isomorphic to C. This cycle generates a frame isomorphic to $\mathfrak{F}$. ☐

Unfortunately, knowing $\mathfrak{G}$ does not yet allow us to find $\mathfrak{F}$. Therefore, we must perform some complex reasoning. Let $lc(\mathfrak{F})$ be the size of the cycle of largest depth in $\mathfrak{F}$ (if that exists; otherwise, this number is zero). We show by induction on $cyc(\mathfrak{F})$ and $lc(\mathfrak{F})$ that $\mathfrak{F}$ is invariant.

Theorem 6.7. *Let $\mathfrak{F}$ be a linear frame. Then $\mathfrak{F}$ is invariant.*

Proof. The proof proceeds by induction on $cyc(\mathfrak{F})$ and $lc(\mathfrak{F})$. The claim proved inductively is the following.

(‡) Let p and q be natural numbers. For all frames $\mathfrak{F}$ such that
(i) $cyc(\mathfrak{F}) < p$ or
(ii) $cyc(\mathfrak{F}) = p$ and $lc(\mathfrak{F}) < q$, $\mathsf{Th}\,\mathfrak{F}$ is invariant.

Put $\langle p, q\rangle \ll \langle p', q'\rangle$ if (i) $p < p'$ or (ii) $p = p'$ and $q < q'$. We show tat the claim holds for $\langle p^*, q^*\rangle$ provided that it holds for all $\langle p, q\rangle \ll \langle p^*, q*\rangle$. The induction starts therefore with $\langle 0, 0\rangle$. By invoking Theorem 4.10 we know that (‡) holds for $\langle 1, 0\rangle$. Notice that if (‡) holds for all $\langle p, q\rangle$ such that $p < p^*$, then it holds for $\langle p^*, q^*\rangle$, by virtue of its definition. Hence, we only need to prove the step from $\langle p, q\rangle$ to $\langle p, q+1\rangle$.

Let $\mathfrak{F}$ be given, with $cyc(\mathfrak{F}) = p$ and $lc(\mathfrak{F}) = q$. We first consider the case where $\mathfrak{F}$ is generated by a cycle. Consider the frame minimal in $\mathbb{S}(\mathfrak{F})$. By Lemma 6.6, we know that $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$. Let C be the leading cycle of $\mathfrak{F}$, $\mathfrak{F}^\circ := \mathfrak{F} \restriction F - C$. By induction hypothesis, $\mathfrak{F}^\circ$ is invariant and can be identified. Let $\mathfrak{H}$ be the largest frame $\geq \mathfrak{G}$ and $\leq \mathfrak{F}^\circ$ such that $cyc(\mathfrak{H}) = cyc(\mathfrak{F})$ and $lc(\mathfrak{H}) = lc(\mathfrak{F})$. This frame is unique. Moreover, it is isomorphic to $\mathfrak{F}$. So, $\mathfrak{F}$ is invariant.

Now we consider the case where $\mathfrak{F}$ is not geneated by a cycle. Let C be the cycle of largest depth, D the largest generated subset of $\mathfrak{F}$ not containing C, and A the complement $F - (C \cup D)$. Then define the following unravellings.

$$\begin{array}{rcl} X(\mathfrak{F}) & := & \langle X(F), \lhd^X\rangle \\ Y(\mathfrak{F}) & := & \langle Y(F), \lhd^Y\rangle \end{array}$$

$$\lhd^X := \begin{cases} & \{\langle\langle y, 0\rangle, \langle c_i, 0, 1\rangle\rangle : y \lhd c_i, y \in A\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle c_{i'}, j+1, 1\rangle\rangle : c_i \lhd c_{i'}\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle c_{i'}, j, 1\rangle\rangle : c_i \lhd c_{i'}, i < i'\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle y, 0\rangle\rangle : c_i \lhd y, y \in E\} \\ \cup & \{\langle\langle x, 0\rangle, \langle\langle y, 0\rangle\rangle : x \lhd y, x, y \in A \cup E\} \end{cases}$$

Further, let

$$\lhd^Y := \lhd^X \cup \{\langle\langle y, 0\rangle, \langle c_i, j, 1\rangle\rangle : y \lhd c_i, j \in \omega\}$$

These define unravellings in the natural way, by restricting to finite parts of the unravelled cycle. These frames have one cycle less than $\mathfrak{F}$, and therefore the induction hypothesis applies to them. This means that they are invariant. Consider the set of frames subreduced by both sets. Denote this set by $\mathbb{T}(\mathfrak{F})$. $\mathfrak{F}$ is contained in this set. Consider an element in this set which is minimal with respect to being $\leq \mathfrak{F}$. By a similar argument as the one above one can show that this element must be isomorphic to $\mathfrak{F}$. □

Corollary 6.8. *Every logic determined by finite linear frames is invariant.*

Since every extension of **S4.3** has the finite model property, we have the following result.

Corollary 6.9. *Every extension of **S4.3** is invariant.*

7. Conclusion

Continuing the research of [3], where it was established that all extensions of **S4.3** are invariant under the automorphisms of the lattice NExt **S4.3**, we have shown here that they are invariant under all automorphisms of NExt **K**. In contrast to that, the lattice NExt $\mathbf{K.alt}_1$ was shown in [3] to have $2^{\aleph_0}$ automorphisms, but here we have established that only the identity extends to an automorphism of NExt **K**, since all extensions of $\mathbf{K.alt}_1$ are invariant. Although this is just a very modest beginning, the present paper shows that completely new techniques must be developed in order to obtain the results. We believe that it can be established with the present methods that all d–homogeneous frames are invariant, and that the unravelling technique allows to extend this result even further. With respect to $\mathbb{P}$–frames in general our intuitions are not so well developed yet.

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