The Semantics of Modal Predicate Logic II.
Modal Individuals Revisited

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Abstract. We continue the investigations begun in [10]. We shall define a semantics that is built on a new notion of frame, called *coherence frames*. In these frames, objects are transcendental (world–independent), as in the standard constant–domain semantics. This will remove the asymmetry between constants and variables of the counterpart semantics of [10]. We demonstrate the completeness of (general) coherence frames with respect to first- and certain weak second–order logics and we shall compare this notion of a frame to counterpart frames as introduced in [10] and the metaframe semantics of [13].

§1. Introduction. In [10] we have developed a semantics that is complete with respect to first- and weak second–order modal predicate logics. This semantics was in addition quite elementary, which was already a great step forward from the previous semantics by Ghilardi [6] and by Skvortsov and Shehtman [13]. Still, from a philosophical point of view this semantics left much to be desired. The introduction of counterpart relations, although in line with at least some philosophical ideas — notably by Lewis —, is not always very satisfactory, since it makes the notion of an object a derived one. The things we see become strictly world bound: there is no sense in which we can talk of, say, the town hall of Berlin, rather than the town hall of Berlin at a particular point of time. The traditional semantics for modal predicate logic held the complete opposite view. Here, objects are transcendental entities. They are not world bound, since they do not belong to the worlds. The difference between these views becomes clear when we look at the way in which the formula $\Diamond\varphi(\vec{x})$ is evaluated. In the standard semantics, we simply go to some accessible world and see whether $\varphi(\vec{x})$ holds. In counterpart semantics, we would not only have to choose another world but also some counterparts for the things that we have chosen as values for the variables in this world. In the traditional semantics the question of counterparts does not arise because of the transcendental status of objects. We may view this as a limiting case of counterpart semantics, in which the counterpart relation always is the identity.

Note that the addition of *constant symbols* to the language introduces further complications. In counterpart semantics, it is far from straightforward to interpret constant symbols, because we need to give an interpretation of these symbols across possible worlds that respects the counterpart relations in some appropriate sense. Variables on the other hand simply denote 'objects' in the domain of a given world. In the case of traditional semantics this asymmetry appears in a similar fashion if one allows constant symbols to be non–rigid, as has been done e.g. in [5]. Then, variables denote transcendental entities, whereas constants denote something like *individual concepts*, i.e functions from possible worlds to a domain. Facing this dilemma, one solution to this is to completely move to a higher–order setting, where constants and variables can be of various higher types, e.g. type–0 constants denote objects, type–1 constants individual concepts etc. (cf. [4]). In this paper, we will follow a different approach, treating constants and variables in the same way, but assuming a more sophisticated notion of a modal individual and identity–at–a–world.

It remains unsatisfactory having to choose between these competing semantics. Moreover, it would be nice if the difference between these semantics was better understood. Certainly, much research has been done into standard semantics and it is known to be highly incomplete if one aims for frame–completeness results.

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However, it is known that completeness with respect to models is as easy to show as in predicate logic but that if the language contains equality, different semantics have to be chosen for different theories/logics of identity (cf. e.g. [7]). The present paper developed from the insight that if the proper semantics is introduced, modal predicate logics with different logics of identity can be treated within the same semantical framework. We call this semantics *coherence semantics.* Completeness with respect to models is then uniformly shown for all modal predicate logics that are extensions of free quantified K together with the predicate logical axioms of equality. We continue by investigating the relationships between coherence frames, counterpart frames and metaframes, discuss the treatment of identity in each of the semantics as well as the interpretation of constant symbols and finally derive a completeness result for so–called cubic generalized metaframes.

§2. Preliminaries. The language has the following symbols. Following Scott [12] we shall work with nonobjectual (possibilist) quantifiers plus an existence predicate. This allows to eliminate the objectual (actualist) quantifiers (they are now definable), and straightens the theory considerably. The existence predicate is a unary predicate whose interpretation — unlike the identity symbol — is completely standard, i.e doesn't have to meet extra conditions. Hence it can actually be suppressed in the notation, making proofs even more simple.

DEFINITION 1 (Symbols and Languages). *The languages of modal predicate logic contain the following symbols.*

1. *A denumerable set* $V := \{x_i : i \in \omega\}$ *of* ***object variables****.*
2. *A denumerable set* $C := \{c_i : i \in \omega\}$ *of* ***constants****.*
3. *A set* Π *of* ***predicate symbols*** *containing the unary* ***existence predicate*** E*.*
4. *The* ***boolean functors*** $\bot$, $\wedge$, $\neg$*.*
5. *The possibilist* ***quantifiers*** $\bigvee$, $\bigwedge$*.*
6. *A set* $M := \{\Box_\lambda : \lambda < \kappa\}$ *of* ***modal operators****.*

Furthermore, each symbol from Π *has an* arity, *denoted by* $\Omega(P)$*. In particular,* $\Omega(E) = 1$*.*

The variables are called x_i, $i \in \omega$. We therefore use x (without subscript!), y, y_j or z, z_k, as metavariables. We assume throughout that we have no function symbols of arity greater than 0. However, this is only a technical simplification. Notice that in [10] we even had no constants. This was so because the treatment of constants in the counterpart semantics is a very delicate affair, which we will discuss below. Moreover, for simplicity we assume that there is only one modal operator, denoted by $\Box$ rather than $\Box_0$. Nothing depends on this choice. The standard quantifiers $\forall$ and $\exists$ are treated as abbreviations.

$$\begin{array}{rcl} (\forall y)\varphi & := & \bigwedge y.E(y) \to \varphi \\ (\exists y)\varphi & := & \bigvee y.E(y) \wedge \varphi \end{array}$$

Moreover, $\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$. The sets of *formulae* and *terms* in this language are built in the usual way. Unless otherwise stated, equality ($\doteq$) is *not* a symbol of the language.

DEFINITION 2 (First–Order MPLs). *A* ***first–order modal predicate logic*** *is a set* L *of formulae satisfying the following conditions.*

1. L *contains all instances of axioms of first–order logic.*
2. L *is closed under all rules of first–order logic.*
3. L *contains all instances of axioms of the modal logic* K*.*
4. L *is closed under the rule* $\varphi/\Box\varphi$*.*
5. $\Diamond \bigvee y.\varphi \leftrightarrow \bigvee y.\Diamond\varphi \in L$*.*

Notice that the last of the postulates ensures that in a Hilbert–style proof all instances of the rule (MN) can be assumed to be at the beginning of the proof. ((MN) 'commutes' with (MP), as is easily seen. However, it commutes with (UG) only in presence of the postulate (5).) To eliminate some uncertainties we shall note that the notions of free and bound occurrences of a variable are exactly the same as in ordinary first–order logic. A variable x occurs **bound** if this occurrence is in the scope of a quantifier $\bigwedge x$ or $\bigvee x$. We denote the simultaneous replacement of the terms s_i for x_i $(i < n)$ in χ by $[s_0/x_0, \dots, s_{n-1}/x_{n-1}]\chi$. Or, writing $\vec{s} = \langle s_i : i < n\rangle$ and $\vec{x} = \langle x_i : i < n\rangle$, we abbreviate this further to $[\vec{s}/\vec{x}]\chi$.

If the language contains equality, the following is required of L.

Eq1. $\bigwedge x.x \doteq x \in L$.
Eq2. $\bigwedge x.\bigwedge y.x \doteq y \to y \doteq x \in L$.
Eq3. $\bigwedge x.\bigwedge y.\bigwedge z.x \doteq y \wedge y \doteq z \to x \doteq z \in L$.
Eq4. $\bigwedge y_0.\bigwedge y_1.\dots\bigwedge y_n.y_i \doteq y_n \to \{P(y_0, \dots, y_{n-1}) \leftrightarrow [y_n/y_i]P(y_0, \dots, y_{n-1})\} \in L$ if $P \in \Pi$, $n = \Omega(P)$.

The axioms Eq1 – Eq3 ensure that equality is interpreted by an equivalence relation. Note that the axiom Eq4 is weaker than the usual Leibniz' Law, because it only allows for the substitutability of identicals in *atomic* predicates. We start with a very basic semantics, standard constant–domain semantics. We assume no equality.

DEFINITION 3 (Frames, Structures and Models). *We call a triple $\mathfrak{W} = \langle W, \lhd, U\rangle$ a **predicate Kripke–frame**, where W is a set (the set of **worlds**), $\lhd \subseteq W \times W$ a binary relation on W (the **accessibility relation**), and U a set (the **universe**). A **modal first–order structure** is a pair $\langle \mathfrak{W}, \mathfrak{I}\rangle$, where $\mathfrak{W}$ is a predicate Kripke–frame and $\mathfrak{I}$ a function mapping a predicate P to a function assigning to each world w an $\Omega(P)$–relation on U and a constant symbol c to a member of U. $\mathfrak{I}$ is called an **interpretation**. Further, $\mathfrak{I}_w$ is the relativized interpretation function at w, which assigns to each $P \in \Pi$ the value $\mathfrak{I}(P)(w)$ and to each constant symbol c the value $\mathfrak{I}(c)$. A **valuation** is a function $\beta : V \to U$. A **model** is a triple $\langle \mathfrak{F}, \beta, w\rangle$ such that $\mathfrak{F}$ is a modal first–order structure, β a valuation into it and w a world of $\mathfrak{F}$.*

As usual, $\gamma \sim_x \beta$ means that $\gamma(y) = \beta(y)$ for all $y \in V$ different from x. If P is a predicate symbol and $\langle t_1, \dots, t_{\Omega(P)}\rangle$ an $\Omega(P)$–tuple of terms, let ε be the function that assigns the tuple $\langle \varepsilon_0(t_0), \dots, \varepsilon_{\Omega(P)-1}(t_{\Omega(P)-1})\rangle$, where $\varepsilon_i = \beta$ if $t_i \in V$ and $\varepsilon = \mathfrak{I}$ if $t_i \in C$.

DEFINITION 4 (Truth in a Model). *Given a modal first–order structure $\mathfrak{F} = \langle \mathfrak{W}, \mathfrak{I}\rangle$, a model $\langle \mathfrak{F}, \beta, w\rangle$, and a formula φ, we define $\langle \mathfrak{F}, \beta, w\rangle \vDash \varphi$ as follows.*

$$\begin{array}{lll}
\langle \mathfrak{F}, \beta, w\rangle \vDash P(\vec{t}) & :\Leftrightarrow & \varepsilon(\vec{t}) \in \mathfrak{I}_w(P) \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \varphi \wedge \chi & :\Leftrightarrow & \langle \mathfrak{F}, \beta, w\rangle \vDash \chi; \varphi \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \neg\varphi & :\Leftrightarrow & \langle \mathfrak{F}, \beta, w\rangle \nvDash \varphi \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \bigvee x.\varphi & :\Leftrightarrow & \textit{for some } \gamma \textit{ with } \gamma \sim_x \beta : \langle \mathfrak{F}, \gamma, w\rangle \vDash \varphi \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \Diamond\varphi & :\Leftrightarrow & \textit{exists } w' \textit{ such that } w \lhd w' \textit{ and } \langle \mathfrak{F}, \beta, w'\rangle \vDash \varphi
\end{array}$$

§3. Completeness. In this section we sketch a proof of the well–known result that every MPL as defined above can be characterized by a canonical structure. The proof presented below is a variation of proofs that can be found in the literature, see e.g. [7], and is based on the use of special, maximal–consistent sets of formulae, namely **Henkin–complete maximal consistent sets** in an extended language. This kind of completeness proof in modal predicate logic goes back to [14]. We present only the basic steps here. But note that the proof depends on the presence of the Barcan formulae.

DEFINITION 5. *A set T of formulae of some language $\mathfrak{L}$ of MPL is called **Henkin–complete**, if for all $\bigvee y.\chi$ in $\mathfrak{L}$ there exists a constant c such that $\bigvee y.\chi \leftrightarrow [c/y]\chi \in T$. Let $\mathfrak{L}^*$ result from a language $\mathfrak{L}$ by adding infinitely many new constant symbols; call $\mathfrak{L}^*$ a **Henkin–language** for $\mathfrak{L}$.*

In what is to follow we will assume that given a language $\mathfrak{L}$, a Henkin–language $\mathfrak{L}^*$ is fixed once and for all.

DEFINITION 6. *Let L be an MPL in $\mathfrak{L}$. A **Henkin–world** is a maximal L–consistent, Henkin–complete set of formulae in the language $\mathfrak{L}^*$.*

LEMMA 7. *Every L–consistent set of formulae in language $\mathfrak{L}$ is contained in some Henkin–world.*

Proof. Let Δ be L–consistent in language $\mathfrak{L}$ and let $\psi_1, \psi_2, \ldots$ be an enumeration of the formulae of type $\bigvee x.\varphi_i(x)$ in the language $\mathfrak{L}^*$. For ψ_i define $\delta_i := \bigvee y.\varphi_i \leftrightarrow [c_i/y]\varphi_i$, where c_i is a new constant not appearing in any δ_j, $j < i$. This is possible because we have an infinite supply of new variables. Define $\Delta^* := \bigcup_{i\in\omega} \Delta_i$, where $\Delta_i := \Delta \cup \{\delta_j : j < i\}$. Then $\Delta_0 = \Delta$. Δ^* is clearly Henkin–complete. By compactness, it is also L–consistent if all Δ_k are. Now, suppose there is a k such that Δ_k is inconsistent. Choose k with this property. By assumption, $k > 0$. There is a finite set $\Delta' \subseteq \Delta$ such that $L \vdash \bigwedge_{\varphi\in\Delta'} \varphi \wedge \bigwedge_{i<k} \delta_i \to \neg\delta_k$. But then $L \vdash \bigwedge_{\varphi\in\Delta'} \varphi \wedge \bigwedge_{i<k} \delta_i \to (\bigvee y.\varphi_k \wedge \neg[c_k/y]\varphi_k)$ where the constant c_k does not appear in any δ_i, $i < k$. Hence, by first–order logic, we have $L \vdash \bigwedge_{\varphi\in\Delta'} \varphi \wedge \bigwedge_{i<k} \delta_i \to (\bigvee y.\varphi_k \wedge \bigwedge y.\neg\varphi_k)$, whence Δ_{k-1} is L–inconsistent, contrary to the choice of k.

Next, by a standard argument, we can turn Δ^* into a maximal L–consistent set in language $\mathfrak{L}^*$ without loosing the property of Henkin–completeness. $\dashv$

Notice that this method of Henkin–closure does not work for the counterpart semantics of [10]. The reason is the asymmetry between variables and constants. Instead, a slightly different definition was used, where instead of constants variables were used as witnesses.

Let C_{L^*} be the set of constant terms of $\mathfrak{L}^*$. Now define W_{L^*} to be the set of all Henkin–worlds. If $\Delta \in W_{L^*}$, the following interpretation is defined.

$$\mathfrak{I}_\Delta(P) := \{\langle c_i : i < \Omega(P)\rangle : P(\vec{c}) \in \Delta\} \text{ and } \mathfrak{I}(c) = c$$

This defines a first–order model on the world Δ. Finally, we put $\Delta \lhd \Sigma$ if for all $\Box\delta \in \Delta$ we have $\delta \in \Sigma$. $\mathfrak{I}_{L^*}$ is defined by piecing the $\mathfrak{I}_\Delta$ together; it assigns to each world Δ the function $\mathfrak{I}_\Delta$. Then we put

$$\mathfrak{Can}_{L^*} := \langle\langle W_{L^*}, \lhd, C_{L^*}\rangle, \mathfrak{I}_{L^*}\rangle$$

This is a modal first–order structure, called the ***canonical structure*** for L. The following is immediate from the definitions.

Lemma 8. *Let Δ be a Henkin–world. Then if $\bigvee y.\chi \in \Delta$, there is a constant d such that $[d/y]\chi \in \Delta$.*

Before we can prove the main result of this section, we need one more lemma whose proof does indeed depend on the presence of the Barcan formulae for the possibilist quantifiers. To state the lemma, let $\Delta^\Box := \{\varphi : \Box\varphi \in \Delta\}$.

Lemma 9. *For every Henkin–world Δ with $\Diamond\chi \in \Delta$, there is a Henkin–world Γ such that $\Delta^\Box \cup \{\chi\} \subset \Gamma$.*

Proof. By a standard argument from propositional modal logic, the set $\Delta^* := \Delta^\Box \cup \{\chi\}$ is L–consistent. We have to show that Δ^* can be extended to a Henkin–world. Note first that this set already contains all the constants from the Henkin–language $\mathfrak{L}^*$. Let $\psi_1, \psi_2, \ldots$ be an enumeration of the formulae of type $\bigvee x.\varphi_i(x)$ in the language $\mathfrak{L}^*$. For ψ_i define $\delta_i := \bigvee y.\varphi_i \leftrightarrow [c_i/y]\varphi_i$, where c_i is the first new constant such that $\Delta_i := \Delta^* \cup \{\delta_j : j < i\}$ is L–consistent. If such a constant always exists we can define Γ as the completion of $\bigcup_{i\in\omega} \Delta_i$ which is a Henkin–world. So suppose that Δ_i is L–consistent but there is no constant c_{i+1} such that Δ_{i+1} is. Then there are, for every constant c of $\mathfrak{L}^*$, formulae $\alpha_0, \ldots, \alpha_{n-1} \in \Delta^\Box$ such that $L \vdash \bigwedge_{k\leq n} \alpha_k \to ((\chi \wedge \bigwedge_{j<i} \delta_i) \to (\bigvee y.\varphi_{i+1} \wedge \neg[c/y]\varphi_{i+1}))$. Since $\Box\alpha_i \in \Delta$ for all i, it follows that $\Box((\chi \wedge \bigwedge_{j<i} \delta_i) \to (\bigvee y.\varphi_{i+1} \wedge \neg[c/y]\varphi_i)) \in \Delta$ for every constant c. Since Δ is Henkin–complete we can 'quantify away' the constant with a variable not appearing in the formula (by using the appropriate Henkin–axiom) and apply the Barcan formula and thus obtain:

$$\Box \bigwedge z.((\chi \wedge \bigwedge_{j<i} \delta_i) \to (\bigvee y.\varphi_i \wedge \neg[z/y]\varphi_i)) \in \Delta.$$

Distributing the quantifier, applying modus tollens and then Box–distribution we arrive at:

$$\Box\neg(\bigvee y.\varphi_i \wedge \bigwedge z.\neg[z/y]\varphi_i)) \to \Box\neg(\chi \wedge \bigwedge_{j<i} \delta_i) \in \Delta.$$

Now, since $\Box(\bigwedge y.\neg\varphi_i \vee \bigvee z.[z/y]\varphi_i)$ belongs to every MPL, we get $\Box\neg(\chi \wedge \bigwedge_{j<i} \delta_i) \in \Delta$, which makes Δ_i inconsistent, contradiction. ⊣

LEMMA 10. *Let φ be a sentence in language $\mathfrak{L}^*$ and Δ a Henkin–world. Then*

$$\langle \mathfrak{Can}_{L^*}, \Delta\rangle \vDash \varphi \Leftrightarrow \varphi \in \Delta$$

Proof. The base case, $\varphi = P(\vec{c})$, follows trivially from the definition of $\mathfrak{I}_\Delta$. The induction steps for $\bot$, $\neg$ and $\wedge$ are routine. Now, let $\varphi = \bigvee y.\chi$. Suppose that $\varphi \in \Delta$. Then, since Δ is Henkin–complete, there exists a constant d such that $[d/x]\chi \in \Delta$. This is a sentence, and by induction hypothesis $\langle \mathfrak{Can}_{L^*}, \Delta\rangle \vDash [d/x]\chi$. Hence, by definition, $\langle \mathfrak{Can}_{L^*}, \Delta\rangle \vDash \bigvee y.\chi$. This argument is reversible. Finally, let $\varphi = \Diamond\chi$. Assume that $\langle \mathfrak{Can}_{L^*}, \Delta\rangle \vDash \Diamond\chi$. Then there exists a Σ such that $\Delta \lhd \Sigma$ and $\langle \mathfrak{Can}_{L^*}, \Sigma\rangle \vDash \chi$. By induction hypothesis, $\chi \in \Sigma$. By definition of $\lhd$, $\Diamond\chi \in \Delta$.

Now assume $\Diamond\chi \in \Delta$. By Lemma 9 there is a Henkin–world Σ such that $\Delta^\Box \cup \{\chi\} \subset \Sigma$. Hence $\Delta \lhd \Sigma$ by definition. By induction hypothesis, $\langle \mathfrak{Can}_{L^*}, \Sigma\rangle \vDash \chi$. So, $\langle \mathfrak{Can}_{L^*}, \Delta\rangle \vDash \Diamond\chi$, as had to be shown. ⊣

Now we have given all the ingredients for a proof of the main result.

THEOREM 11. *Every modal predicate logic without equality is complete with respect to modal first–order structures, in particular*

$$\mathfrak{Can}_{L^*} \vDash \varphi \textit{ iff } \varphi \in L.$$

§4. Coherence Structures. Let us now see what happens if equality is introduced into the language. Evidently, if equality is just a member of Π instead of being a logical symbol, the previous proofs go through. Then the interpretation of equality is an equivalence relation in each world. But generally one requires that equality must be interpreted as identity. Nonetheless, we must ask: identity of what? Think about the example of Hesperus and Phosphorus. As for the real world they are identical, but there are some people for whom they are not. Let George be such a person. Then there is a belief world of George's in which Hesperus and Phosphorus are not identical. Many have argued that George's beliefs are inconsistent. This is what comes out if we assume standard semantics. But we could turn this around in the following way. We say that equality does not denote identity of *objects* but of something else, which we shall call the *object trace*. We say that Hesperus and Phosphorus are different objects, which happen to have the same trace in this world, but nonidentical traces in each of George's belief worlds. To make this distinction between object and object trace more acceptable we shall give a different example. Suppose someone owns a bicycle b and he has it repaired. The next day he picks it up; but then it has a different front wheel. Surely, he would consent to the statement that the bicycle he now has is that bicycle that he gave to the repair shop yesterday. But its front wheel isn't. Let's assume for simplicity that atoms are permanent, they will never cease to exist nor come into existence. Next, let us assume (again simplifying things considerably) that the trace of an object is just the collection of atoms of which it consists. Then, while the object b continued to exist, its trace has changed from one day to the other. In order not to get confused with the problem of transworld identity let us stress that we think of the objects as transcendental. b is neither a citizen of this world today nor of yesterday's world, nor of any other world. But its trace in this world does belong to this world. We may or may not assume that object traces are shared across worlds. Technically matters are simpler if they are not, but nothing hinges on that. So, in addition to the bicycle b we have two wheels w and w', and the trace of b contained the object trace of w yesterday, and it contains the trace of w' today. In the light of these examples it seems sensible to distinguish an object from its trace. Of course, we are not committed to any particular view of traces and certainly do not want to assume that object traces are simply conglomerates of atoms.

Now, in the classical semantics, identity across worlds was a trivial matter. Objects were transcendental, and in using the same letter we always refer to the same object across worlds. However, identity is not relative to worlds. If Hesperus is the same object as Phosphorus in one world, it is the same in all worlds. The distinction between object and trace gets us around this problem as follows. Denote the objects by h and p; further, let this world be w_0 and let w_1 be one of George's belief worlds. Then the traces of h and p are the same in this world, but different in w_1. This solves the apparent problem. In our words, equality does not denote identity of two objects, but only identity of their traces in a particular world.

DEFINITION 12 (Coherence Frames and Structures). *By a* ***coherence frame*** *we understand a quintuple* $\langle W, \lhd, U, T, \tau\rangle$ *where* $\langle W, \lhd, U\rangle$ *is a predicate Kripke–frame,* T *a set, the set of* ***things****, and* $\tau : U \times W \to T$ *a surjective function. We call* τ *the* ***trace function*** *and* $\tau(o, w)$ *the* ***trace of*** o ***in*** w*. An* ***interpretation*** *is a function* $\mathfrak{I}$ *mapping each* $P \in \Pi$ *to a function from* W *to* $U^{\Omega(P)}$ *and each constant symbol* c *to a member of* U*.* $\mathfrak{I}$ *is called* ***equivalential*** *if for all* $\vec{a}, \vec{b} \in U^{\Omega(P)}$ *and* $w \in W$*, if* $\tau(a_i, w) = \tau(b_i, w)$ *for all* $i < \Omega(P)$ *then* $\vec{a} \in \mathfrak{I}(P)(w)$ *iff* $\vec{b} \in \mathfrak{I}(P)(w)$*. A* ***coherence structure*** *is a pair* $\langle \mathfrak{W}, \mathfrak{I}\rangle$ *where* $\mathfrak{W}$ *is a coherence frame and* $\mathfrak{I}$ *an equivalential interpretation.*

Note that since trace functions are assumed to be surjective, every trace has to be the trace of some object. This is a natural condition, because objects are considered to be the primary entity, and traces a derived notion. The notion of equivalence is perhaps a curious one. It says that the basic properties of objects cannot discriminate between objects of equal trace. So, if Pierre believes that London is beautiful and Londres is not, we have two objects which happen to have the same trace in this world. Hence they must share all properties *in this world.* So, London and Londres can only be both beautiful or both ugly. This seems very plausible indeed. From a technical point of view, however, the fact that they cannot simply have different properties is a mere stipulation on our part. On the other hand, it is conceivable that there are basic predicates that are actually intensional, which would mean that they fail the substitution under (extensional) equality.

An alternative setup for strictly extensional basic predicates is the following. An interpretation is a function assigning to predicates in a world not tuples of objects but tuples of things. Then an object has a property iff its trace does. This approach is certainly more transparent because it attributes the fact that an object bears a property only to the fact that its trace does. Yet, technically it amounts to the same.

DEFINITION 13 (Coherence Models). *A* ***coherence model*** *is a triple* $\langle \mathfrak{C}, \beta, w\rangle$*, where* $\mathfrak{C}$ *is a coherence structure,* $\beta : V \to U$ *a valuation,* $w \in W$ *and* ε *as in Definition 4. We define the truth of a formula inductively as follows.*

$$\begin{array}{lll}
\langle \mathfrak{C}, \beta, w\rangle \vDash P(\vec{t}) & :\Leftrightarrow & \varepsilon(\vec{t}) \in \mathfrak{I}_w(P) \\
\langle \mathfrak{C}, \beta, w\rangle \vDash s \doteq t & :\Leftrightarrow & \tau(\varepsilon(s), w) = \tau(\varepsilon(t), w) \\
\langle \mathfrak{C}, \beta, w\rangle \vDash \chi \wedge \varphi & :\Leftrightarrow & \langle \mathfrak{C}, \beta, w\rangle \vDash \chi; \varphi \\
\langle \mathfrak{C}, \beta, w\rangle \vDash \neg\varphi & :\Leftrightarrow & \langle \mathfrak{C}, \beta, w\rangle \nvDash \varphi \\
\langle \mathfrak{C}, \beta, w\rangle \vDash \bigvee x.\varphi & :\Leftrightarrow & \textit{for some } \gamma \textit{ with } \gamma \sim_x \beta : \langle \mathfrak{C}, \gamma, w\rangle \vDash \varphi \\
\langle \mathfrak{C}, \beta, w\rangle \vDash \Diamond\varphi & :\Leftrightarrow & \textit{there is } w' \textit{ such that } w \lhd w' \textit{ and } \langle \mathfrak{C}, \beta, w'\rangle \vDash \varphi
\end{array}$$

$\mathfrak{C} \vDash \varphi$ *if for all valuations* β *and all worlds* v*:* $\langle \mathfrak{C}, \beta, v\rangle \vDash \varphi$*.*

It is a matter of straightforward verification that all axioms and rules of the minimal MPL are valid in a coherence frame. Moreover, the set of formulae valid in a coherence structure constitute a first–order MPL. Notice that the fourth postulate for equality holds in virtue of the special clause for equality and the condition that the interpretation must be equivalential. For if $\langle \mathfrak{C}, \beta, w\rangle \vDash y_i \doteq y_n$, then $\tau(\beta(y_i), w) = \tau(\beta(y_n), w)$. So, if $\langle \mathfrak{C}, \beta, w\rangle \vDash P(y_0, \dots, y_{n-1})$ for $P \in \Pi$, then $\langle \beta(y_i) : i < n\rangle \in \mathfrak{I}(P)(w)$. Let $\beta' \sim_{y_i} \beta$ be such that $\beta'(y_i) = \beta(y_n)$. By equivalentiality, $\langle \beta'(y_i) : i < n\rangle \in \mathfrak{I}(P)(w)$. This means that $\langle \mathfrak{C}, \beta', w\rangle \vDash P(y_0, \dots, y_{n-1})$, and so $\langle \mathfrak{C}, \beta, w\rangle \vDash [y_n/y_i]P(y_0, \dots, y_{n-1})$. If $\mathfrak{F}$ is a coherence frame, put $\mathfrak{F} \vDash \varphi$ if $\langle \mathfrak{F}, \mathfrak{I}\rangle \vDash \varphi$ for all equivalential interpretations $\mathfrak{I}$. Evidently, $\{\varphi : \mathfrak{F} \vDash \varphi\}$ is a first–order MPL.

The difference with the counterpart semantics is that we have disentangled the quantification over objects from the quantification over worlds. Moreover, objects exist independently of worlds. Each object leaves a trace in a given world, though it need not exist there. Furthermore, two objects can have the same trace in any given world without being identical. However, identity of two objects holds in a world iff they have the same trace in it. If we also have function symbols, the clauses for basic predicates and equality will have to be generalized in the obvious direction.

To derive a completeness result for coherence structures we have to revise the construction from Section 3 only slightly, namely we have to define what the traces of objects are. To do this, let $\Delta \in W_{L^*}$ be a Henkin–world and c a constant. Then put $[c]_\Delta := \{d \in C_{L^*} : c \doteq d \in \Delta\}$. Now set

$$\tau_{L^*}(c, \Delta) := \langle [c]_\Delta, \Delta\rangle.$$

Then let

$$T_{L^*} := \{\langle [c]_\Delta, \Delta\rangle : \Delta \in W_{L^*}, c \in C_{L^*}\}.$$

Finally, put

$$\mathfrak{Coh}_{L^*} := \langle\langle W_{L^*}, \lhd, C_{L^*}, T_{L^*}, \tau_{L^*}\rangle, \mathfrak{I}_{L^*}\rangle.$$

This is a coherence structure. For by Eq4, $\mathfrak{I}_{L^*}$ is equivalential, as is easily checked. Since $\langle [c]_\Delta, \Delta\rangle = \langle [d]_\Delta, \Delta\rangle$ iff $[c]_\Delta = [d]_\Delta$ iff $c \doteq d \in \Delta$, the following is immediate:

Lemma 14. *Let φ be a sentence and Δ a Henkin–world. Then*

$$\langle \mathfrak{Coh}_{L^*}, \Delta\rangle \vDash \varphi \Leftrightarrow \varphi \in \Delta\ .$$

Theorem 15. *Every modal predicate logic with or without equality is complete with respect to coherence structures.*

In his [11], Gerhard Schurz introduced a semantics, called *worldline semantics*, in the context of analyzing Hume's is–ought thesis, i.e. the logical problem whether one may infer ethical value (normative) statements from factual (descriptive) statements. This semantics is very close to the coherence semantics defined in this paper. A **worldline frame** is a quintuple $\langle W, R, L, U, Df\rangle$, where W is a set or worlds, R the accessibility relation, $U \neq \varnothing$ a non–empty set of **possible objects**, $\varnothing \neq L \subseteq U^W$ a set of functions from possible worlds to possible objects (members of L are called **worldlines**), $Df : W \to \wp(U)$ a **domain function** such that $Df(w) =: D_w \subseteq U_w$, where $U_w := \{d \in U : \exists l \in L(l(w) = d)\}$ (the set of term extensions at world w) and $L_w = \{l \in L : \exists d \in D_w(l(w) = d)\}$ (the set of worldlines with extension in D_w). An interpretation V into a worldline frame is a function such that $V(t) \in L$ for any term t and $V_w(Q) \subseteq U_w^n$ for any n–ary predicate Q. If V is a worldline interpretation denote by $V[l/x]$ the interpretation that is like V except that it assigns worldline l to the variable x. Since, unlike Schurz, we assume that free logical quantifiers are a defined notion, we suppose in the following that $D_w = U_w$ for all w. Then, in particular, $Df(w) = U_w$ and $L_w = L$ for all w, which means that Df can be omitted. The truth relation in worldline semantics can now be defined as follows:

Definition 16 (Truth in Worldline Semantics). *Let $\mathfrak{F} = \langle W, R, L, U\rangle$ be a worldline frame, V an interpretation and w a world, define*

$$\begin{array}{lll}
\langle \mathfrak{F}, V, w\rangle \vDash P(\vec{t}) & :\Leftrightarrow & V_w(\vec{t}) \in V_w(P) \\
\langle \mathfrak{F}, V, w\rangle \vDash s \doteq t & :\Leftrightarrow & V(s)(w) = V(t)(w) \\
\langle \mathfrak{F}, V, w\rangle \vDash \chi \wedge \varphi & :\Leftrightarrow & \langle \mathfrak{F}, V, w\rangle \vDash \chi; \varphi \\
\langle \mathfrak{F}, V, w\rangle \vDash \neg\varphi & :\Leftrightarrow & \langle \mathfrak{F}, V, w\rangle \nvDash \varphi \\
\langle \mathfrak{F}, V, w\rangle \vDash \bigvee x.\varphi & :\Leftrightarrow & \textit{for some } l \in L : \langle \mathfrak{F}, V[l/x], w\rangle \vDash \varphi \\
\langle \mathfrak{F}, V, w\rangle \vDash \Diamond\varphi & :\Leftrightarrow & \textit{there is } w' \textit{ such that } w \lhd w' \textit{ and } \langle \mathfrak{F}, V, w'\rangle \vDash \varphi
\end{array}$$

It should be rather clear that the main difference between worldline and coherence semantics is terminological. While in worldline semantics one quantifies over worldlines and evaluates predicates and identity statements with respect to the value of a worldline at a particular world, in coherence semantics we quantify over modal individuals without specifying their internal structure, but assume a trace function that maps an individual at a world to its trace. So we can give the following translation. Given a coherence model $\langle \mathfrak{F}, \mathfrak{I}, \beta, w\rangle$ based on the coherence frame $\langle W, \lhd, U, T, \tau\rangle$, define a worldline model $\langle \mathfrak{G}, V, w'\rangle$ based on the worldline frame $\langle W', R, L, U'\rangle$ as follows. Set $W' := W$, $R := \lhd$, $U' := T$, and $w' := w$. Further, given $u \in U$, define $f_u : W' \to U'$ by letting $f_u(w) := t$ if $\tau(u, w) = t$ and set $L := \{f_u : u \in U\}$. Then, for $v \in W'$, we have

$U'_v := \{t \in U' : \exists l \in L(l(v) = t)\} = \{t \in T : \exists u \in U(\tau(u,v) = t)\}$. Call $\mathfrak{G}$ the **worldline companion** of $\mathfrak{F}$.

PROPOSITION 17. *For every coherence frame $\mathfrak{F}$ and its worldline companion $\mathfrak{G}$ and for all φ:*

$$\mathfrak{F} \vDash \varphi \textit{ iff } \mathfrak{G} \vDash \varphi$$

In particular, if a logic L is frame complete with respect to coherence frames, it is frame complete with respect to worldline frames.

Proof. The proof is by a rather straightforward structural induction on φ. The only task is to define appropriate interpretations, so we only consider the atomic cases. Fix a coherence frame $\mathfrak{F}$ and its worldline companion $\mathfrak{G}$. Suppose $\varphi = P(x_0, \dots, x_{n-1})$ and that $\langle \mathfrak{F}, \mathfrak{I}, \beta, w\rangle \nvDash \varphi$ for some equivalential interpretation $\mathfrak{I}$, valuation β and world w. Define

$$\langle t_0, \dots t_{n-1}\rangle \in V_w(P) \text{ iff } t_i = \tau(w, u_i) \text{ and } \langle u_0, \dots u_{n-1}\rangle \in \mathfrak{I}_w(P) \qquad (*)$$

and $\beta_w(x_i) = V_w(x_i)$. Then, clearly, $\langle \mathfrak{G}, V, w\rangle \nvDash P(x_0, \dots x_{n-1})$. Conversely, given an interpretation V and $\langle \mathfrak{G}, V, w\rangle \nvDash \varphi$, we can define an interpretation $\mathfrak{I}$ and a valuation β as in $(*)$ such that $\mathfrak{I}$ is equivalential and $\langle\langle \mathfrak{F}, \mathfrak{I}\rangle, \beta, w\rangle \nvDash \varphi$. The case $\varphi = (x_i \doteq x_j)$ is treated in the same way. $\dashv$

§5. Coherence Structures and Counterpart Structures. Counterpart frames and structures were introduced in Kracht and Kutz [10]. They generalize the functor–semantics of Ghilardi. Call a relation $R \subseteq M \times N$ a **CE–relation** (CE stands for 'counterpart existence') if for all $x \in M$ there exists a $y \in N$ such that $x \mathrel{R} y$ and, likewise, for all $y \in N$ there exists an $x \in M$ such that $x \mathrel{R} y$. This is a slight adaptation of the definition of that paper to take care of the fact that we now deal with possibilist quantifiers plus an existence predicate as opposed to 'proper' free–logical quantifiers. Furthermore, we shall make more explicit the world dependence of the universes.

DEFINITION 18 (Counterpart Frames and Structures). *A quadruple $\langle W, T, \mathcal{U}, \mathcal{C}\rangle$ is a* ***counterpart frame*** *if $W, T \neq \varnothing$ are non–empty sets, $\mathcal{U}$ a function assigning to each member v of W a non–empty subset $\mathcal{U}_v$ of T (its* ***domain****) and, finally, $\mathcal{C}$ a function assigning to each pair of worlds v, w a set $\mathcal{C}(v,w)$ of CE–relations from $\mathcal{U}_v$ to $\mathcal{U}_w$. A pair $\langle \mathfrak{W}, \mathfrak{I}\rangle$ is called a* ***counterpart structure*** *if $\mathfrak{W}$ is a counterpart frame and $\mathfrak{I}$ an interpretation, that is, a function assigning to each $w \in W$ and to each n–ary predicate letter a subset of $\mathcal{U}_w^n$.*

We say that v **sees** w in $\mathfrak{F}$ if $\mathcal{C}(v,w) \neq \varnothing$. A **valuation** is a function η which assigns to every possible world v and every variable an element from the universe $\mathcal{U}_v$ of v. We write η_v for the valuation η at v. A **counterpart model** is a quadruple $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{I}, \eta, w\rangle$, where $\mathfrak{F}$ is a counterpart frame, $\mathfrak{I}$ an interpretation, η a valuation and $w \in W$. Note that interpretations in counterpart frames differ from interpretations in coherence frames in that they don't assign values to constants. That is to say, unless otherwise stated, when working with counterpart frames we assume that the language does not contain constants.

Let $v, w \in W$ be given and ρ a CE–relation from $\mathcal{U}_v$ to $\mathcal{U}_w$. We write $\eta \xrightarrow{\rho} \widetilde{\eta}$ if for all $x \in V$: $\langle \eta_v(x), \widetilde{\eta}_w(x)\rangle \in \rho$. In the context of counterpart frames, $\widetilde{\eta} \sim_x^v \eta$ denotes a local x–variant at the domain of world v, i.e., $\widetilde{\eta}$ is a valuation that may differ from η only in the values that it assigns to the variable x at world v.

DEFINITION 19 (Truth in a Counterpart Model). *Let $\varphi(\vec{y})$ and $\psi(\vec{z})$ be modal formulae with the free variables $y_0, \dots, y_{n-1}$ and $z_0, \dots, z_{m-1}$, respectively. Let $\mathfrak{C}$ be a counterpart structure, v a* *possible world and let η be a valuation. We define:*

$$\begin{array}{lll}
\langle \mathfrak{C}, \eta, v\rangle \vDash x_i \doteq x_j & :\Leftrightarrow & \eta_v(x_i) = \eta_v(x_j) \\
\langle \mathfrak{C}, \eta, v\rangle \vDash R(\vec{y}) & :\Leftrightarrow & \langle \eta_v(y_0), \ldots, \eta_v(y_{n-1})\rangle \in \mathfrak{I}_v(R) \\
\langle \mathfrak{C}, \eta, v\rangle \vDash \neg\varphi & :\Leftrightarrow & \langle \mathfrak{C}, \eta, v\rangle \nvDash \varphi \\
\langle \mathfrak{C}, \eta, v\rangle \vDash \varphi \wedge \chi & :\Leftrightarrow & \langle \mathfrak{C}, \eta, v\rangle \vDash \varphi; \chi \\
\langle \mathfrak{C}, \eta, v\rangle \vDash \Diamond\varphi(\vec{y}) & :\Leftrightarrow & \text{there are } w \in W, \rho \in \mathcal{C}(v,w) \text{ and } \widetilde{\eta} \text{ such that } \eta \xrightarrow{\rho} \widetilde{\eta} \text{ and} \\
& & \qquad \langle \mathfrak{C}, \widetilde{\eta}, w\rangle \vDash \varphi(\vec{y}) \\
\langle \mathfrak{C}, \eta, v\rangle \vDash \bigvee x.\varphi(x) & :\Leftrightarrow & \text{there is } \widetilde{\eta} \sim_x^v \eta \text{ such that } \langle \mathfrak{C}, \widetilde{\eta}, v\rangle \vDash \varphi(x)
\end{array}$$

Given a counterpart frame $\mathfrak{F}$, $\mathfrak{F} \vDash \varphi$ if for all interpretations $\mathfrak{I}$, all valuations η and worlds v, $\langle\langle \mathfrak{F}, \mathfrak{I}\rangle, \eta, v\rangle \vDash \varphi$.

The intuition behind counterpart frames is that objects do not exist; the only things that exist are the object traces (which belong to the domains of the worlds), and the counterpart relations. However, the notion of an object is still definable, even though it shall turn out that counterpart frames can have very few objects in this sense.

DEFINITION 20. *Let $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C}\rangle$ be a counterpart frame. An **object** is a function $f : W \to T$ such that (i) $f(v) \in \mathcal{U}_v$ for all $v \in W$, (ii) for each pair $v, w \in W$ with $\mathcal{C}(v,w) \neq \varnothing$ there is $\rho \in \mathcal{C}(v,w)$ such that $\langle f(v), f(w)\rangle \in \rho$.*

So, objects are constructed using the counterpart relation. If the trace b in world w is a counterpart of the trace a in world v, then there may be an object leaving trace a in v and trace b in w. If not, then not. However, there are frames which are not empty and possess no objects. Here is an example. Let $W = \{v\}$, $T = \{a, b\}$, $\mathcal{U}_v = \{a, b\}$, and $\mathcal{C}(v, v) = \{\rho\}$ with $\rho = \{\langle a, b\rangle, \langle b, a\rangle\}$. It is easy to see that this frame has no objects. The crux is that we can only choose one trace per world, but when we pass to an accessible world, we must choose a counterpart. This may become impossible the moment we have cycles in the frame.

Counterpart frames show a different behaviour than coherence frames. As we have shown above, for each modal predicate logic there exists an adequate structure. However, counterpart structures satisfy a formula that is actually not generally valid when one thinks of the quantifiers as quantifying over intensional rather than extensional (trace–like) objects.

PROPOSITION 21. *Let $\mathfrak{F}$ be a counterpart frame, x and y variables not occurring in $\vec{z}$. Then for all formulae $\varphi(x, \vec{z})$*

$$\mathfrak{F} \vDash \bigwedge x. \bigwedge y.(x \doteq y) \to (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z}))$$

Proof. Pick a valuation $\mathfrak{I}$, and let $\mathfrak{S} := \langle \mathfrak{F}, \mathfrak{I}\rangle$. It is clear that we can restrict our attention to formulae of the type $\varphi(x, \vec{z}) = \Diamond\chi(x, \vec{z})$. Let η be a valuation and v a world. Assume that $\langle \mathfrak{S}, \eta, v\rangle \vDash x \doteq y$. Then $\eta_v(x) = \eta_v(y)$. We will show that $\langle \mathfrak{S}, \eta, v\rangle \vDash \Diamond\chi(x, \vec{z}) \to \Diamond\chi(y, \vec{z})$. Suppose therefore that $\langle \mathfrak{S}, \eta, v\rangle \vDash \Diamond\chi(x, \vec{z})$. Then there exists a world w, a $\rho \in \mathcal{C}(v, w)$ and a valuation $\widetilde{\eta}$ such that $\eta \xrightarrow{\rho} \widetilde{\eta}$ and $\langle \mathfrak{S}, \widetilde{\eta}, w\rangle \vDash \chi(x, \vec{z})$. Now define η' by $\eta'_w(y) := \widetilde{\eta}_w(x)$, and $\eta'_{w'}(y') := \widetilde{\eta}_{w'}(y')$ for all w' and y' such that either $w' \neq w$ or $y' \neq y$. Then $\langle \mathfrak{S}, \eta', w\rangle \vDash \chi(y, \vec{z})$. Furthermore, for all variables y': $\langle \eta_v(y'), \eta'_w(y')\rangle \in \rho$. For if $y' \neq y$ this holds by definition of η and choice of $\widetilde{\eta}$. And if $y' = y$ we have $\eta_v(x) = \eta_v(y)$, so that $\rho \ni \langle \eta_v(x), \widetilde{\eta}_w(x)\rangle = \langle \eta_v(y), \eta'_w(y)\rangle$. It follows that $\eta \xrightarrow{\rho} \eta'$ and therefore that $\langle \mathfrak{S}, \eta, v\rangle \vDash \Diamond\chi(y, \vec{z})$, as had to be shown. ⊣

From the previous theorem we deduce that also the following holds.

$$(\ddagger) \qquad \bigwedge x. \bigwedge y.(x \doteq y \wedge \Diamond\top) \to \Diamond(x \doteq y)$$

Namely, take

$$\bigwedge x. \bigwedge y. x \doteq y. \to .\Diamond(x \doteq z) \leftrightarrow \Diamond(y \doteq z)$$

By the above theorem, this is generally valid. Substituting x for z we get

$$\bigwedge x. \bigwedge y. x \doteq y. \to .\Diamond(x \doteq x) \leftrightarrow \Diamond(y \doteq x)$$

Applying standard laws of predicate logic yields (‡). We remark here that the logics defined in the literature (for example Ghilardi [6], Skvortsov and Shehtman [13] and Kracht and Kutz [10]), differ from modal predicate logics as defined here only in the additional laws of equality that they assume.

The modal Leibniz law of [10] allows for simultaneous substitution of all free occurrences of x by y in $\Diamond\chi$ (denoted by $\Diamond\chi(y//x)$, provided that $x \doteq y$ is true.

$$\bigwedge x. \bigwedge y. x \doteq y. \to .\Diamond\chi(x) \to \Diamond\chi(y//x)$$

Now, notice that in a modal predicate logic as defined above, the rule of replacing constants for universally quantified variables is valid. In counterpart frames this creates unexpected difficulties. For, suppose we do have constants and that they may be substituted for variables. Then we may derive from (‡), using the substitution of c for x and d for y:

$$(c \doteq d \wedge \Diamond\top) \to \Diamond(c \doteq d)$$

Since a constant has a fixed interpretation in each world, this means that if two constants are equal in a world and there exists some accessible possible world, then there will also be some accessible world in which they are equal. This is not generally valid. What is happening here is a shift from a *de re* to a *de dicto* interpretation. If we follow the traces of the objects, the formula is valid, but if we substitute intensional objects, namely constants, it becomes refutable. Notice that this situation is also reflected in the way, non–rigid constants are treated in [5]. There, the two possible readings of the above formula, the de dicto and de re reading, are distinguished by actually binding the interpretation of the constants to the respective worlds by using the term–binding λ–operator.

Applied to Hesperus and Phosphorus, this means that if they are equal, then there is a belief world of George's in which they are equal. However, if George believes that they are different, this cannot be the case. So, the counterpart semantics cannot handle constants correctly—at least not in a straightforward way, i.e., without restricting the possible values of constants in accessible worlds. This paradox is avoided in Kracht and Kutz [10] by assuming that the language actually has *no* constants.

§6. Objectual Counterpart Structures. The connection between coherence frames and counterpart frames is not at all straightforward. Since the logic of a counterpart frame is a first–order modal predicate logic, one might expect that for every counterpart frame there is a coherence frame having the same logic. This is only approximately the case. We will show in Theorem 32 that for every counterpart *structure* there is a coherence structure having the same theory. This is not generally true for frames. However, adopting a modification of coherence frames proposed by Melvin Fitting in [3], namely balanced coherence frames (in [3] the corresponding frames are called Riemann FOIL frames), it can indeed be shown that for every counterpart frame there is a balanced coherence frame with the same theory.

Let us begin by elucidating some of the connections between counterpart and coherence frames. Note again that since counterpart structures as defined above don't interpret constants, we have to assume that the language does not contain constants.

First, fix a coherence structure $\mathfrak{C} = \langle W, \lhd, U, T, \tau, \mathfrak{I}\rangle$. We put $U_v := \{\tau(o, v) : o \in U\}$. This defines the domains of the world. Next, for $v, w \in W$ we put $\rho(v, w) := \{\langle\tau(o, v), \tau(o, w)\rangle : o \in U\}$ and $\mathcal{C}(v, w) := \varnothing$ if $v \lhd w$ does not obtain; otherwise, $\mathcal{C}(v, w) := \{\rho(v, w)\}$. Finally, $\langle\tau(a_i, w) : i < \Omega(P)\rangle \in \mathfrak{I}'(P)(w)$ iff $\langle a_i : i < \Omega(P)\rangle \in \mathfrak{I}(P)(w)$. Then $\langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{I}'\rangle$ is a counterpart structure. We shall denote it by $CP(\mathfrak{C})$. Notice, that there is at most one counterpart relation between any two worlds.

Conversely, let a counterpart structure $\mathfrak{N} = \langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{I}\rangle$ be given. We put $v \lhd w$ iff $\mathcal{C}(v, w) \neq \varnothing$. $U := T$. Let O be the set of all objects $o : W \to T$. Further, $\tau(o, w) := o(w)$. This defines a coherence frame if the set of objects is nonempty. Finally, $\langle o_i : i < \Omega(P)\rangle \in \mathfrak{I}'(P)(w)$ iff $\langle o_i(w) : i < n\rangle \in \mathfrak{I}(P)(w)$. It is easy to see that this is an equivalential interpretation. So, $\langle W, \lhd, O, U, \tau, \mathfrak{I}'\rangle$ is a coherence structure, which we denote by $CH(\mathfrak{N})$.

Unfortunately, the logical relation between these two types of structures is rather opaque, not the least since the notion of satisfaction in them is different. Moreover, the operations just defined are not inverses of each other. For example, as we have already seen, there exist counterpart structures with nonempty domains which have no objects. In this case $CP(CH(\mathfrak{N})) \ncong \mathfrak{N}$. Also let $\mathfrak{C}$ be the following coherence frame. $W := \{v, w, x, y\}$, $T := \{1, 2, 3, 4, 5, 6\}$, $U = \{a, b\}$, $\lhd = \{\langle v, w\rangle, \langle w, x\rangle, \langle x, y\rangle\}$. Finally, $\tau(a, -) : v \mapsto 1, w \mapsto 2, x \mapsto 4, y \mapsto 5$, $\tau(b, -) : v \mapsto 1, w \mapsto 3, x \mapsto 4, y \mapsto 6$. Generating the counterpart frame we find that 2 and 3 are counterparts of 1, and 5 and 6 are counterparts of 4. Hence, there are more objects in the counterpart frame than existed in the coherence frame, for example the function $v \mapsto 1, w \mapsto 2, x \mapsto 4, y \mapsto 6$.

Definition 22 (Threads). *Let $\mathfrak{N}$ be a counterpart frame. A sequence $\langle (w_i, t_i) : i < n\rangle$ is called a* ***thread*** *if (1) for all $i < n$: $w_i \in W$, $t_i \in \mathcal{U}_{w_i}$, and (2) for all $i < n - 1$: $w_i \lhd w_{i+1}$ and $\langle t_i, t_{i+1}\rangle \in \rho$ for some $\rho \in \mathcal{C}(w_i, w_{i+1})$. $\mathfrak{N}$ is* ***rich in objects*** *if for all threads there exists an object o such that $o(w_i) = t_i$ for all $i < n$.*

Notice that if $\lhd$ is cycle free then $\mathfrak{N}$ is automatically object rich. Otherwise, we must only be able to come back to the same thing when going in a cycle. This is a rather strict condition. Nevertheless, we can use unravelling to produce an equivalent cycle free structure from a given one, which is then object rich. Additionally, we can ensure that between any two worlds there is at most one counterpart relation. We call counterpart frames that satisfy the condition $|\mathcal{C}(v, w)| \leq 1$ for all worlds $v, w \in W$ **Lewisian counterpart frames**.

Theorem 23. *For every counterpart structure $\mathfrak{N}$ there exists a Lewisian counterpart structure $\mathfrak{N}'$ rich in objects such that $\mathfrak{N}$ and $\mathfrak{N}'$ have the same theory.*

Proof. Let $\mathfrak{N} = \langle\langle W, T, \mathcal{U}, \mathcal{C}\rangle, \mathfrak{I}\rangle$ be a counterpart structure. A **path** in $\mathfrak{N}$ is a sequence $\pi = \langle w_0, \langle\langle w_i, \rho_i\rangle : 0 < i < n\rangle\rangle$ such that $\rho_i \in \mathcal{C}(w_{i-1}, w_i)$ for all $0 < i < n$. We let $e(\pi) := w_{n-1}$ and $r(\pi) = \rho_{n-1}$ and call these, respectively, the **end point** and the **end relation** of π. Let W' be the set of all paths in $\mathfrak{N}$ and $T' := T$. Further, let $\mathcal{U}'_\pi := \mathcal{U}_{e(\pi)}$ and for two paths π and μ put $\mathcal{C}'(\pi, \mu) := r(\mu)$ if $r(\mu) \in \mathcal{C}(e(\pi), e(\mu))$ and empty otherwise. Finally, let P be an n–ary predicate letter. Then $\mathfrak{I}'(P)(\pi) := \mathfrak{I}(P)(e(\pi))$. Now let $\mathfrak{N}' = \langle\langle W', T', \mathcal{U}', \mathcal{C}'\rangle, \mathfrak{I}'\rangle$. This is a Lewisian counterpart structure and clearly rich in objects. The following can be verified by induction. If β is a valuation on $\mathfrak{N}$, and w a world, and if β' is a valuation on $\mathfrak{N}'$ and π a path such that $e(\pi) = w$ and $\beta'_\pi(x_i) = \beta_w(x_i)$, then $\langle \mathfrak{N}, \beta, w\rangle \vDash \varphi$ iff $\langle \mathfrak{N}', \beta', \pi\rangle \vDash \varphi$ for all φ. The theorem now follows; for given β and w, β' and π satisfying these conditions can be found, and given β' and π also β and w satisfying these conditions can be found. $\dashv$

Notice by the way that in the propositional as well as the second–order case this theorem is false. This is so because the interpretation of a predicate in π must be identical to that of μ if the two have identical end points. If this is not the case, the previous theorem becomes false. However, if we are interested in characterizing MPLs by means of models, it follows from the above result that we can restrict ourselves in the discussion to Lewisian counterpart structures that are rich in objects.

But we can also strike the following compromise. Let us keep the counterpart semantics as it is, but interpret formulae in a different way. Specifically, let us define the following.

Definition 24 (Objectual Counterpart Interpretations). *We say that $\mathfrak{M} = \langle\langle \mathfrak{C}, \mathfrak{I}\rangle, \beta, v\rangle$ is an* ***objectual counterpart model****, if $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C}\rangle$ is a counterpart frame as before, $\mathfrak{I}$ is an* ***objectual interpretation****, that is, a counterpart interpretation that additionally assigns objects to constant symbols, β an* ***objectual valuation*** *into $\mathfrak{F}$, i.e., a function that assigns to each variable an object in a given a world. In this context, $\varepsilon_v(o) := \beta_v(o)$ if o is a variable and $\varepsilon_v(o) = \mathfrak{I}(o)(v)$ if o is a constant symbol.*

Write $\beta \to^{\vec{y}}_{v,w} \beta$ if for some $\rho \in \mathcal{C}(v, w)$ we have $\langle \beta_v(x_i), \beta_w(x_i)\rangle \in \rho$ for all $x_i \in \vec{y}$. Furthermore, write $\beta \to^{\vec{y}}_{v,w} \gamma$ if for some $\rho \in \mathcal{C}(v, w)$ we have $\langle \beta_v(x_i), \gamma_w(x_i)\rangle \in \rho$ for all $x_i \in \vec{y}$, where γ is an objectual valuation. Terms t_i denote either variables or constants, $\vec{y}$ tuples of variables and $\vec{c}$

tuples of constants. The symbol $\vDash^*$ *is called the* ***weak objectual truth–relation*** *and is defined thus (with* $\mathfrak{M} := \langle \mathfrak{F}, \mathfrak{I} \rangle$*):*

$$\begin{array}{lll}
\langle \mathfrak{M}, \beta, v \rangle \vDash^* t_i \doteq t_j & :\Leftrightarrow & \varepsilon_v(t_i) = \varepsilon_v(t_j) \\
\langle \mathfrak{M}, \beta, v \rangle \vDash^* R(\vec{t}) & :\Leftrightarrow & \langle \varepsilon_v(t_0), \ldots, \varepsilon_v(t_{n-1}) \rangle \in \mathfrak{I}_v(R) \\
\langle \mathfrak{M}, \beta, v \rangle \vDash^* \neg\varphi & :\Leftrightarrow & \langle \mathfrak{M}, \beta, v \rangle \nvDash^* \varphi \\
\langle \mathfrak{M}, \beta, v \rangle \vDash^* \varphi \wedge \chi & :\Leftrightarrow & \langle \mathfrak{M}, \beta, v \rangle \vDash^* \varphi; \chi \\
\langle \mathfrak{M}, \beta, v \rangle \vDash^* \Diamond\varphi(\vec{y}, \vec{c}) & :\Leftrightarrow & \textit{there is } \beta \to^{\vec{y}}_{v,w} \gamma \textit{ such that } \langle \mathfrak{M}, \gamma, w \rangle \vDash^* \varphi(\vec{y}, \vec{c}) \\
\langle \mathfrak{M}, \beta, v \rangle \vDash^* \bigvee y.\varphi(y, \vec{c}) & :\Leftrightarrow & \textit{there is } \widetilde{\beta} \sim_y \beta \textit{ such that } \langle \mathfrak{M}, \widetilde{\beta}, v \rangle \vDash^* \varphi(y, \vec{c})
\end{array}$$

The ***strong objectual truth–relation*** $\vDash^\dagger$ *is like* $\vDash^*$ *except for the clause for* $\Diamond$ *which is given by:*

$$\langle \mathfrak{M}, \beta, v \rangle \vDash^\dagger \Diamond\varphi(\vec{y}, \vec{c}) \quad :\Leftrightarrow \quad \textit{there is } \beta \to^{\vec{y}}_{v,w} \beta \textit{ and } \langle \mathfrak{M}, \beta, w \rangle \vDash^\dagger \varphi(\vec{y}, \vec{c})$$

These interpretations remove the asymmetry between variables and constants in the sense that constants and variables are now assigned the same kind of values. However, while the strong objectual interpretation brings us very close to coherence semantics, the weak interpretation still bears essential properties of counterpart semantics, namely that we may move via a counterpart relation to a new object. More precisely we have the following:

PROPOSITION 25. *The rule of substituting constants for universally quantified variables is valid in the strong objectual interpretation. More specifically, for every counterpart frame* $\mathfrak{F}$

$$\mathfrak{F} \vDash^\dagger (\bigwedge x.\varphi) \to [c/x]\varphi.$$

Furthermore, there is an objectual counterpart model $\mathfrak{M}$ *such that*

$$\mathfrak{M} \nvDash^\dagger \bigwedge x. \bigwedge y.(x \doteq y) \to (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z}))$$

Both claims are false for the weak objectual interpretation.

Proof. For the first claim suppose that β is an objectual valuation, $\mathfrak{I}$ an objectual interpretation, v a world and that $\langle \langle \mathfrak{F}, \mathfrak{I} \rangle, \beta, v \rangle \vDash^\dagger \bigwedge x.\varphi$. We only need to consider the case where $\varphi = \Diamond\psi$. We then have that for every x–variant $\widetilde{\beta}$: $\langle \langle \mathfrak{F}, \mathfrak{I} \rangle, \widetilde{\beta}, v \rangle \vDash^\dagger \Diamond\psi(x)$. I.e., for every object $o = \widetilde{\beta}(x)$ there is $\widetilde{\beta} \to^x_{v,w} \widetilde{\beta}$ and $\langle \langle \mathfrak{F}, \mathfrak{I} \rangle, \widetilde{\beta}, w \rangle \vDash^* \psi(x)$. Now $\mathfrak{I}(c) = \widetilde{\beta}(x)$ for some x–variant $\widetilde{\beta}$ of β from which the claim follows immediately.

For the second claim, fix the following simple model $\mathfrak{M}$. Let $W = \{v, w\}$, $T = \{a, b, b'\}$, $\mathfrak{U}(v) = \{a\}$, $\mathfrak{U}(w) = \{b, b'\}$, $\mathfrak{C}(v, w) = \{\rho\}$ where $\rho = \{\langle a, b \rangle, \langle a, b' \rangle\}$ and $\mathfrak{I}(P)(w) = \{b\}$ and $\beta(x) = o$ with $o(v) = a$ and $o(w) = b$ and $\beta(y) = o'$ with $o'(v) = a$ and $o'(w) = b'$. o and o' are the only objects in this model. It should be obvious that $\mathfrak{M} \vDash^\dagger x \doteq y \wedge \Diamond P(x)$ while $\mathfrak{M} \nvDash^\dagger \Diamond P(y)$.

Consider now the weak objectual interpretation. Take the model just defined and assume furthermore that $\mathfrak{I}(c) = o'$. Then clearly $v \vDash^* \bigwedge x.\Diamond P(x)$ while $w \nvDash^* \Diamond P(c)$, which shows that the rule is not valid. That the formula in the second claim is still valid under the weak objectual interpretation should be clear. ⊣

The following is also straightforward.

THEOREM 26. *Let* $\mathfrak{N}$ *be a counterpart structure rich in objects,* v *a world and let* β *be an objectual valuation and* $\widetilde{\beta}$ *a counterpart valuation such that* $\beta_v(x_i) = \widetilde{\beta}_v(x_i)$ *for all variables. Then for all* φ*:*

$$\langle \mathfrak{N}, \beta, v \rangle \vDash^* \varphi \quad \Leftrightarrow \quad \langle \mathfrak{N}, \widetilde{\beta}, v \rangle \vDash \varphi$$

The proof is by induction on φ. The two relations differ only with respect to formulae of the form $\Diamond\chi$. Here, object richness assures that for each choice of counterparts in the successor worlds an object exists. (Actually, for that we only need that every element of a domain is the trace of some object.)

§7. Passing Back and Forth. By the previous theorem we can introduce the notion of an object into counterpart frames, which then makes them rather similar to coherence frames. However, counterpart structures with object valuations are still different from coherence structures. A different approach is to translate $\Diamond$ in order to accommodate the truth relation $\vDash^*$ within the language of counterpart structures.

$$\begin{array}{lcl}
(x \doteq y)^\gamma & := & x \doteq y \\
P(\vec{y})^\gamma & := & P(\vec{y}) \\
(\neg\varphi)^\gamma & := & \neg\varphi^\gamma \\
(\varphi \wedge \chi)^\gamma & := & \varphi^\gamma \wedge \chi^\gamma \\
(\bigvee x.\varphi)^\gamma & := & \bigvee x.\varphi^\gamma \\
(\Diamond\varphi(y_0, \dots, y_{n-1}))^\gamma & := & \bigvee z_0 . \dots \bigvee z_{n-1} . \bigwedge_{i<n} z_i \doteq y_i \wedge \Diamond\varphi(\vec{z}/\vec{y})^\gamma
\end{array}$$

Here, y_i ($i < n$) are the free variables of φ and z_i ($i < n$) distinct variables not occurring in φ. This is actually unique only up to renaming of bound variables. Further, notice that $\bigwedge_{i<n}$ denotes a finite conjunction, not a quantifier. This translation makes explicit the fact that variables inside a $\Diamond$ are on a par with bound variables. (In linguistics, one speaks of $\Diamond$ in the context of counterpart frames as an *unselective binder*.) Notice now that

$$\begin{array}{ll}
 & (\bigwedge x_0 . \bigwedge x_1 . x_0 \doteq x_1 \to (\Diamond\varphi(x_0) \leftrightarrow \Diamond\varphi(x_1)))^\gamma \\
= & \bigwedge x_0 . \bigwedge x_1 . x_0 \doteq x_1 \to ((\Diamond\varphi(x_0))^\gamma \leftrightarrow (\Diamond\varphi(x_1))^\gamma) \\
= & \bigwedge x_0 . \bigwedge x_1 . x_0 \doteq x_1 \to ((\bigvee x_3)(x_3 \doteq x_0 \wedge \Diamond\varphi(x_3)^\gamma) \leftrightarrow ((\bigvee x_3)(x_3 \doteq x_1 \wedge \Diamond\varphi(x_3)^\gamma))
\end{array}$$

This principle is actually valid in coherence structures. For it is a substitution instance of the following theorem of predicate logic.

$$\bigwedge x_0 . \bigwedge x_1 . x_0 \doteq x_1 \to ((\bigvee x_3 . x_3 \doteq x_0 \wedge \varphi) \leftrightarrow (\bigvee x_3 . x_3 \doteq x_1 \wedge \varphi))$$

Proposition 27. *Let $\mathfrak{N}$ be a counterpart structure and x a world. Then for any φ:*

$$\langle \mathfrak{N}, x \rangle \vDash \varphi^\gamma \quad \Leftrightarrow \quad \langle \mathfrak{N}, x \rangle \vDash \varphi$$

In object rich structures also $\vDash$ and $\vDash^*$ coincide, which makes all four notions the same. So, while in counterpart structures formulae φ and φ^γ are equivalent, they are certainly not equivalent when interpreted in coherence structures.

In [11] it is shown that worldline semantics provides for the same class of frame complete logics in the absence of extra equality axioms as standard constant domain semantics. It follows that the same holds for coherence frames. This means that while coherence frames allow for a more natural treatment of non–rigid designation for example, unlike counterpart frames, they don't enlarge the class of frame complete logics unless one moves to the full second–order semantics as we will do in Section 10. But there is a different approach to this problem. Instead of introducing algebras of admissible interpretations we can assume that certain worlds are isomorphic copies of each other. So, we add to the frames an equivalence relation between worlds and require that predicates are always interpreted in the same way in equivalent worlds. This idea is basically due to Melvin Fitting (see [3]). Here we use a slightly different approach. Namely, we add a first–order bisimulation to coherence frames. To be precise, let $\mathfrak{F}$ be a coherence frame. Call a relation $\mathcal{E} \subseteq W \times W$ a **world–mirror on** $\mathfrak{F}$ if $\mathcal{E}$ is an equivalence relation and whenever $v \,\mathcal{E}\, w$ and $v \lhd u_1$, there is a u_2 such that $w \lhd u_2$ and $u_1 \,\mathcal{E}\, u_2$. Intuitively, two mirrored worlds v and w may be understood as a situation seen from two different perspectives (because v and w may have 'different histories', but have the 'same future'). In [9] world–mirrors are called **nets**, and it is shown that an equivalence relation is a net iff it is induced by a p–morphism.

Definition 28 (Balanced Coherence Frames). *A* ***balanced coherence frame*** *is a pair $\langle \mathfrak{F}, \mathcal{E} \rangle$ where $\mathfrak{F} = \langle W, \lhd, U, T, \tau \rangle$ is a coherence frame and $\mathcal{E}$ is a world–mirror on $\mathfrak{F}$. An interpretation $\mathfrak{I}$ is called* ***balanced****, if it is equivalential and $\langle u_0, \dots, u_{n-1} \rangle \in \mathfrak{I}_v(P)$ iff $\langle u_0, \dots, u_{n-1} \rangle \in \mathfrak{I}_w(P)$ for all n–ary relations P and for all worlds v, w such that $v \,\mathcal{E}\, w$. A* ***balanced coherence model*** *is a triple $\langle\langle \mathfrak{B}, \mathfrak{I} \rangle, \beta, w \rangle$, where $\mathfrak{B}$ is a balanced coherence frame, $\mathfrak{I}$ a balanced interpretation, β a valuation and w a world.*

The next theorem gives the connection between counterpart frames and balanced coherence frames.

THEOREM 29. *For every counterpart frame $\mathfrak{F}$ there exists a balanced coherence frame $\mathfrak{F}^*$ such that for all formulae φ:*

$$\mathfrak{F} \vDash \varphi \quad \Leftrightarrow \quad \mathfrak{F}^* \vDash \varphi^\gamma.$$

Proof. Fix a counterpart frame $\mathfrak{F} = \langle W, T, \mathfrak{U}, \mathfrak{C}\rangle$. Let $\mathfrak{F}' := \langle W', T', \mathfrak{U}', \mathfrak{C}'\rangle$ be the unravelled Lewisian counterpart frame defined as in Theorem 23. We define a balanced coherence frame $\mathfrak{F}^* = \langle\langle W^*, \lhd, U^*, T^*, \tau\rangle, \mathcal{E}\rangle$ from $\mathfrak{F}'$. Let $W^* = W'$, $T^* = T'$ and define $\pi \lhd \nu :\Leftrightarrow \mathfrak{C}'(\pi, \nu) \neq \varnothing$. Since $\mathfrak{F}'$ is rich in objects, there is an object $o : W' \to T$ for every thread in $\mathfrak{F}'$. Define U^* as the set of all objects in $\mathfrak{F}'$ and set $\tau(o, w) = t :\Leftrightarrow o(w) = t$. Finally, for $\pi, \nu \in W^*$, set $\pi\, \mathcal{E}\, \nu :\Leftrightarrow e(\pi) = e(\nu)$, where $e(\pi), e(\nu)$ again denote the endpoints of $e(\pi)$ and $e(\nu)$, respectively. Clearly, $\mathcal{E}$ is a world–mirror for it is an equivalence relation and if $\pi \lhd \mu$ and $\pi\, \mathcal{E}\, \nu$, there is a path μ' such that $r(\mu) = r(\mu')$ and $\nu \lhd \mu'$, hence $\mu\, \mathcal{E}\, \mu'$.

Now suppose that $\mathfrak{F} \nvDash \varphi$. Then there is a valuation β, an interpretation $\mathfrak{I}$ and a world v such that $\langle\langle \mathfrak{F}, \mathfrak{I}\rangle, \beta, v\rangle \nvDash \varphi$. By Theorem 23, $\langle\langle \mathfrak{F}', \mathfrak{I}'\rangle, \beta', \nu\rangle \nvDash \varphi$, where $\mathfrak{I}'(P)(\pi) := \mathfrak{I}(P)(e(\pi))$ for all worlds π, $\beta'_\pi(x_i) := \beta_w(x_i)$ if $e(\pi) = w$ and ν is a world in $\mathfrak{F}'$ such that $e(\nu) = v$. Set $\mathfrak{I}^*(P)(\pi) := \{\langle o_0, \dots, o_{n-1}\rangle \in (U^*)^n : \langle o_0(e(\pi)), \dots, o_{n-1}(e(\pi))\rangle \in \mathfrak{I}'(P)(\pi)\}$ and choose an object valuation β^* on $\mathfrak{F}^*$ such that $\beta^*(x_i)(\nu) = \beta'(x_i)$. Such a valuation exists because $\mathfrak{F}'$ is rich in objects. Furthermore, $\mathfrak{I}^*$ is a balanced interpretation, so $\langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \beta^*, \nu\rangle$ defines a balanced coherence model. We claim that $\mathfrak{M}^* = \langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \beta^*, \nu\rangle \nvDash \varphi^\gamma$. The atomic and Boolean cases are trivial and the quantificational case follows from object richness. So, consider the case $\varphi = \Diamond\psi(y_0, \dots, y_{n-1})$. We have to show that $\mathfrak{M}^* \vDash \bigwedge z_0 \dots \bigwedge z_{n-1}.(\bigwedge_{i<n} z_i \doteq y_i \to \Box\neg\psi(\vec{z}/\vec{y})^\gamma)$. Choose objects $o_0, \dots, o_{n-1}$ and a $\vec{z}$–variant $\widetilde{\beta^*}$ such that $o_i(\pi) = \widetilde{\beta^*_\pi}(z_i) = \beta^*_\pi(y_i)$ for all $i < n$. Then $\langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \widetilde{\beta^*}, \nu\rangle \vDash \bigwedge_{i<n} z_i \doteq y_i$. Assume further that for some $\pi \in W^*$ we have $\nu \lhd \pi$, i.e. that there is $\rho \in \mathfrak{C}'(\nu, \pi)$. Then, since $\mathfrak{F}'$ is a Lewisian counterpart frame, ρ is unique and hence $\langle o_i(\nu), o_i(\pi)\rangle \in \rho$ for all $i < n$. Then, by assumption, we have $\langle\langle \mathfrak{F}, \mathfrak{I}\rangle, \widetilde{\beta}, w\rangle \nvDash \psi$, where $e(\pi) = w$ and $\widetilde{\beta_\pi}(z_i) = \widetilde{\beta^*_\pi}(z_i)$, so by induction it follows that $\langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \widetilde{\beta^*}, \pi\rangle \vDash \neg\psi(\vec{z}/\vec{y})^\gamma$.

Conversely, suppose that $\mathfrak{F}^* \nvDash \varphi^\gamma$. Again we consider only the case of

$$\varphi^\gamma = \bigvee z_0 \dots . \bigvee z_{n-1}(\bigwedge_{i<n} z_i \doteq y_i \wedge \Diamond\psi(\vec{z}/\vec{y})^\gamma)$$

and assume that we can find a countermodel for ψ if a countermodel for ψ^γ is given. Pick a balanced interpretation $\mathfrak{I}^*$, an object valuation β^* and a world ν such that $\langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \beta^*, \nu\rangle \nvDash \varphi^\gamma$. Let $\mathfrak{M}'$ be the Lewisian counterpart model $\langle\langle \mathfrak{F}', \mathfrak{I}'\rangle, \beta', \nu\rangle$, where $\mathfrak{F}'$ is as above,

$$\mathfrak{I}'(P)(\pi) = \{\langle t_0, \dots, t_{n-1}\rangle \in (T')^n : \exists o_i \in U^* \text{ s.t. } o_i(\pi) = t_i (i < n) \text{ and } \langle o_0, \dots o_{n-1}\rangle \in \mathfrak{I}^*(P)(\pi)\}$$

for all worlds π and $\beta'_\nu(y_i) = \beta^*(y_i)(\nu)$. We claim that $\mathfrak{M}' \vDash \Box\neg\psi$, which implies by Theorem 23 that $\mathfrak{F} \nvDash \varphi$. So, suppose there is a world π such that $\nu \lhd \pi$ in $\mathfrak{F}'$, i.e. that there is $\delta \in \mathfrak{C}(e(\nu), e(\pi))$ and $\delta = r(\pi)$. Suppose $\widetilde{\beta}'$ is a counterpart valuation such that $\widetilde{\beta}' : \beta' \xrightarrow{\delta} \widetilde{\beta}'$. By object richness there are objects $u_i \in U^*(i < n)$ such that $u_i(\nu) = \beta'_\nu(y_i)$ and $u_i(\pi) = \widetilde{\beta}'_\pi(y_i)$. Hence there is an objectual $\vec{z}$–variant $\widetilde{\beta^*}$ such that $\widetilde{\beta^*}(z_i)(\nu) = \beta^*(y_i)(\nu)$ for all i. Then $\langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \widetilde{\beta^*}, \nu\rangle \vDash \bigwedge_{i<n} z_i \doteq y_i$ whence $\langle\langle \mathfrak{F}^*, \mathfrak{I}^*\rangle, \widetilde{\beta^*}, \pi\rangle \vDash \neg\psi^\gamma(\vec{z}/\vec{y})$. But $\widetilde{\beta^*}(z_i)(\pi) = \widetilde{\beta}'_\pi(y_i)$ and the claim follows. $\dashv$

This result has interesting consequences. For example, since counterpart semantics is frame complete with respect to all first–order extensions QL of canonical propositional modal logics L (compare [6]), the same holds true for the translation QL^γ with respect to balanced coherence frames. Now we noted above that coherence frames per se characterize the same logics as standard constant domain semantics if no extra equality axioms are involved. But it is known that already rather simple canonical propositional logics have frame incomplete predicate extensions. In [2] it is shown that to complete frame incomplete MPLs by adding appropriate axioms, one needs mixed de re formulae rather than substitution instances of purely propositional formulae. So, the above result gives a hint on where the source for frame incompleteness with respect to standard semantics is to be found. In particular, note that the translation $.^{\gamma}$ leaves propositional formulae untouched, whereas de re formulae of the form $\Diamond\varphi(y_0, \dots, y_{n-1})$ are transformed into formulae $\bigvee z_0 . \dots \bigvee z_{n-1} . \bigwedge_{i<n} z_i \doteq y_i \wedge \Diamond\varphi(\vec{z}/\vec{y})^{\gamma}$, which are de re formulae involving equality. So what we need if we want to use standard possible worlds semantics to characterize a large class of logics via frame completeness are two things: firstly, a different understanding of the modal operator as given by $.^{\gamma}$, and, secondly, the assumption that certain worlds are copies of each other.

We shall also use balanced frames to give a proof of another result. It follows from a simple observation. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be coherence frames. A **p–morphism** from $\mathfrak{C}$ to $\mathfrak{C}'$ is a function $h : W \to W'$ such that h is a p–morphism from $\langle W, \lhd\rangle$ to $\langle W', \lhd'\rangle$, $U' = U$, $T' = T$, and $\tau'(o, \pi(v)) = \tau(o, v)$ for all $o \in U$ and $v \in W$.

Proposition 30. *Let $\langle\mathfrak{C}, \mathcal{E}\rangle$ and $\langle\mathfrak{C}', \mathcal{E}'\rangle$ be balanced coherence frames and $h : \mathfrak{C} \to \mathfrak{C}'$ be a p–morphism. If $h[\mathcal{E}] = \mathcal{E}'$ then for all φ:*

$$\langle\langle\mathfrak{C}, \mathcal{E}\rangle, v\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle\langle\mathfrak{C}', \mathcal{E}'\rangle, h(v)\rangle \vDash \varphi$$

The proof is straightforward. For the next theorem we need another auxiliary technique which goes as follows. Let $\mathfrak{C} := \langle W, \lhd, U, T, \tau\rangle$ be a coherence frame. Put $\mathfrak{C}^{\times} := \langle W, \lhd, U \times V, T, \tau^{\times}\rangle$, where $\tau^{\times}(\langle o, x\rangle, w) := \tau(o, w)$. Next let γ be a valuation into $\mathfrak{C}$. Then let $\gamma^{\times}(x) := \langle\gamma(x), x\rangle$. It is easy to see that

$$\langle\mathfrak{C}, \gamma, w\rangle \vDash \varphi \qquad \Leftrightarrow \qquad \langle\mathfrak{C}^{\times}, \gamma^{\times}, w\rangle \vDash \varphi$$

From this fact it easily follows:

Lemma 31. *For every w and every φ: $\langle\mathfrak{C}, w\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle\mathfrak{C}^{\times}, w\rangle \vDash \varphi$. Moreover, if $\langle\mathfrak{C}^{\times}, w\rangle \nvDash \varphi$, there is a valuation γ which is injective on the free variables of φ such that $\langle\mathfrak{C}^{\times}, \gamma, w\rangle \vDash \neg\varphi$.*

We call a frame **crowded** if every formula not valid at a world w can be refuted at w using valuation which is injective on the free variables of φ.

Theorem 32. *For every balanced coherence frame $\mathfrak{B} = \langle\mathfrak{C}, \mathcal{E}\rangle$ there exists a balanced coherence frame $\mathfrak{B}^{+} = \langle\mathfrak{C}^{+}, \mathcal{E}^{+}\rangle$ such that for every world x of $\mathfrak{C}$ there exists a world y of $\mathfrak{C}^{+}$ such that for any formula φ*

$$\langle\langle\mathfrak{C}, \mathcal{E}\rangle, x\rangle \vDash \varphi^{\gamma} \quad \Leftrightarrow \quad \langle\langle\mathfrak{C}^{+}, \mathcal{E}^{+}\rangle, y\rangle \vDash \varphi$$

Proof. In view of Lemma 31 we may actually assume that $\mathfrak{C}$ is crowded. Let v be a world. A function $f : U \to U$ is called v–**admissible** if $\tau(f(o), v) = \tau(o, v)$ for all $o \in U$. A γ–sequence is a sequence $v_0; f_0; v_1; f_1; \dots ; f_{n-1}; v_n$ such that for all $i < n$: f_i is v_i–admissible and $v_i \lhd v_{i+1}$. Then let W^{+} be the set of all γ–sequences of $\mathcal{C}$. Put $H \lhd^{+} H'$ iff there are J, v, g and w such that $H = J; v$, $H' = J; v; g; w$ and $w \rhd v$. Put $H\, \mathcal{E}^{+}\, H'$ iff for some J, J', v and v': $H = J; v$, $H' = J'; v'$, and $v\, \mathcal{E}\, v'$. $U^{+} := U$, $T^{+} := T$. Finally, $\tau^{+}(o, H) := \tau((f_{n-1} \circ f_{n-2} \dots \circ f_0)(o), v_n)$. This finishes the definition of the balanced coherence frame. Note that $\xi : H; id_U; v \mapsto H; f; v$ is a bijective map from the transit of $H; id_U; v$ onto the transit of $H; f; v$ and if $\gamma'(x) = f(\gamma(x))$ for every free variable of χ then

$$\langle\langle\mathfrak{B}^{+}, \mathfrak{I}\rangle, \gamma', J\rangle \vDash \chi \quad \Leftrightarrow \quad \langle\langle\mathfrak{B}^{+}, \mathfrak{I}\rangle, \gamma, \xi(J)\rangle \vDash \chi$$

The map $\zeta : J; v \mapsto v$ that sends a γ–sequence to its end point is a p–morphism of the Kripke–frames, and $h^{-1}[\mathcal{E}] = \mathcal{E}^{+}$. Let $\gamma : V \to U$ be a valuation $\mathfrak{I}$ a balanced interpretation in $\mathcal{E}^{+}$, φ a formula without equation, and $H \in W^{+}$. $\mathfrak{I}$ induces a balanced interpretation $\mathfrak{I}_{+}$ on $\mathfrak{B}$. We show that if γ is injective on the free variables of φ then

$$(\ddagger) \qquad \langle\langle\mathfrak{B}^{+}, \mathfrak{I}\rangle, \gamma, H\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle\langle\mathfrak{B}, \mathfrak{I}_{+}\rangle, \gamma, \zeta(H)\rangle \vDash \varphi^{\gamma}$$

Since every balanced interpretation $\mathfrak{J}$ on $\mathfrak{B}$ is of the form $\mathfrak{I}_{+}$ for some interpretation $\mathfrak{I}$ on $\mathfrak{B}^{+}$, the above claim holds for every balanced interpretation, and so we get from $(\ddagger)$ that

$$\langle\mathfrak{B}^{+}, \gamma, H\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle\mathfrak{B}, \gamma, \zeta(H)\rangle \vDash \varphi^{\gamma}$$

Finally, observe that any x is the image under ζ of some H. This gives the claim.

(‡) is shown by induction. Since $\cdot^{\gamma}$ commutes with the boolean operations and the quantifiers, we only need to look at the diamond and the base clauses. $\mathfrak{I}_+(P)(\zeta(H)) = \{\vec{o} : \vec{o} \in \mathfrak{I}(P)(H)\}$, by construction. Thus, $\langle\langle \mathfrak{B}^+, \mathfrak{I}\rangle, \gamma, H\rangle \vDash P(\vec{s})$ iff $\langle \varepsilon(s_i) : i < \Omega(P)\rangle \in \mathfrak{I}(P)(H)$ iff $\langle \varepsilon(s_i) : i < \Omega(P)\rangle \in \mathfrak{I}_+(P)(\zeta(H))$ iff $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma, \zeta(H)\rangle \vDash P(\vec{s})$.

The only remaining case is $\varphi = \Diamond\chi(\vec{x})$, $\vec{x} = x_0, \ldots, x_{n-1}$. Then $(\Diamond\chi(\vec{x}))^{\gamma} = \bigvee \vec{y}.\vec{y} \doteq \vec{x} \wedge \Diamond\chi(\vec{y})^{\gamma}$. Let $v := \zeta(H)$ for some H (which can be arbitrary). $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma, \zeta(H)\rangle \vDash (\Diamond\chi(x))^{\gamma}$ iff for some $\gamma' \sim_{\vec{y}} \gamma$ such that $\tau(\gamma'(y_i), v) = \tau(\gamma(x_i), v)$ for all $i < n$ we have $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma', v\rangle \vDash \Diamond\chi(\vec{y})^{\gamma}$. Moreover, using that $\mathfrak{C}$ is crowded, one can show that there is a γ' which is injective on the free variables of this formula. So there is a $w \rhd v$ such that $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma', w\rangle \vDash \chi(\vec{y})^{\gamma}$. By inductive hypothesis, $\langle\langle \mathfrak{B}^+, \mathfrak{I}\rangle, \gamma', H; id_U; w\rangle \vDash \chi(\vec{y})$. Let f be defined as follows. $f(\gamma(x_i)) := \gamma'(y_i)$, and $f(o) := o$ if $o \neq \gamma(x_i)$ for all $i < n$. Since γ is injective on $\vec{x}$, this is well defined. Furthermore, f is injective on the set $\{\gamma(x_i) : i < n\}$, by injectivity of γ'. We show that f is v–admissible. (i) If $o \neq \gamma(x_i)$ for all $i < n$, $\tau(f(o), v) = \tau(o, v)$, since $f(o) = o$. (ii) For some i, $o = \gamma(x_i)$. This i is unique given o. Then $\tau(f(\gamma(x_i)), v) = \tau(\gamma'(y_i), v)$, by definition of f. Also, $\tau(\gamma'(y_i), v) = \tau(\gamma(x_i), v)$, showing that $\tau(f(o), v) = \tau(o), v)$, as required. Using the observation on isomorphisms above, $\langle\langle \mathfrak{B}^+, \mathfrak{I}\rangle, \gamma, H; f; w\rangle \vDash \chi(\vec{x})$. Now, $H \lhd^+ H; f; w$, and so $\langle\langle \mathfrak{B}^+, \mathfrak{I}\rangle, \gamma, H\rangle \vDash \Diamond\chi(\vec{x})$, which had to be shown. Assume for the converse that $\langle\langle \mathfrak{B}^+, \mathfrak{I}\rangle, \gamma, H\rangle \vDash \Diamond\chi(\vec{x})$. Then for some v–admissible f and $w \rhd v$, $\langle\langle \mathfrak{B}^+, \mathfrak{I}\rangle, \gamma, H; f; w\rangle \vDash \chi(\vec{x})$. By inductive hypothesis, $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma, w\rangle \vDash \chi(\vec{x})^{\gamma}$ and so $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma, v\rangle \vDash \Diamond(\chi(\vec{x})^{\gamma})$, and, therefore, $\langle\langle \mathfrak{B}, \mathfrak{I}_+\rangle, \gamma, v\rangle \vDash \bigvee \vec{y}.\vec{y} \doteq \vec{x} \wedge \Diamond\chi(\vec{y}/\vec{x})^{\gamma}$.

Let us see what happens if φ contains occurrences of $\doteq$. If these are not inside a modal operator it is easily seen that the claim still holds. (It holds for all choices of points $H = v_0; id_U; v_1; id_U; v_2; \ldots$.) So, let us look at the occurrences of equations inside a modal operator. Bring φ into a form where in every subformulae $\Diamond\chi$ or $\neg\Diamond\chi$, χ is a conjunction of prime formulae or formulae of the form $(\neg)\Diamond\xi$. These manipulations commute with $\cdot^{\gamma}$. Now look at a subformula $\Diamond\chi(\vec{x})$. If it contains equations, these are of the form $x_i \doteq x_j$ for certain i, j, or $x_i \doteq c$ for some constant c. Then $(\Diamond\chi(\vec{x}))^{\gamma}$ contains equations of the form $y_i \doteq y_j$, $y_i \doteq c$ inside $\Diamond$ and $y_i \doteq x_i$ outside. Then choose γ' as above. We have $\tau(\gamma'(y_i), w) = \tau(\gamma'(y_j), w)$ if $y_i \doteq y_j$ is a conjunct of χ, and $\tau(\gamma'(y_i), w) = \tau(\mathfrak{I}_+(c), w)$ if $y_i \doteq c$ is a conjunct. f is defined as above. Then if $\tau(\gamma'(y_i), w) = \tau(\gamma'(y_j), w)$ we will get $\tau(f(\gamma(y_i)), w) = \tau(f(\gamma(y_j)), w)$. The equations are therefore true by choice of the function f. The rest now follows as before by induction. ⊣

Using the same construction one can also show the following.

THEOREM 33. *For every balanced coherence frame* $\mathfrak{B}$ *there exists a balanced coherence frame* $\mathfrak{B}^+$ *such that* $\{\varphi^{\gamma} : \mathfrak{B} \vDash \varphi^{\gamma}\} = \{\varphi : \mathfrak{B}^+ \vDash \varphi\}$.

§8. Varieties of Equality. Scott [12] proposes various kinds of identity. The first is the one we have discussed so far, namely **identity in trace**. The second, stronger notion, is the **inherent identity in trace**, which we shall denote by $\doteq^{\star}$. Two objects satisfy this at a world if they are identical in trace at all subsequent worlds. The third is the **global identity in trace**, which we denote by $\approx$. Two objects are globally identical in trace at w if their traces are identical in all worlds that can be reached from w by either moving forward or backward along the relations. The fourth is **strong identity in trace**, denoted by $\approx^+$. Two objects are strongly identical in trace if they have identical trace in all worlds. The fifth is **identity as object**, denoted by $\equiv$. This is the numerical identity of objects. The semantics can be formally defined as follows. Denote by $T(w)$ the set of all worlds which are accessible from w in a series of steps. More formally, we define this as follows.

DEFINITION 34 (Transits). *Let* $\mathfrak{C} = \langle W, \lhd, U, T, \tau\rangle$ *be a coherence frame. Then define* $v \lhd^n w$ *inductively by (a)* $v \lhd^0 w$ *iff* $v = w$, *(b)* $v \lhd^{n+1} w$ *iff there is a* $u \in W$ *such that* $v \lhd^n u \lhd w$. *Further, put* $v \lhd^* w$ *iff there is an* n *such that* $v \lhd^n w$. *Define* $T(v) := \{w : v \lhd^* w\}$, *and* $Z(v) := \{w : v(\lhd \cup \lhd^{\smile})^* w\}$.

Here, $R^{\smile} := \{\langle y, x\rangle : \langle x, y\rangle \in R\}$ is the converse relation of R.

DEFINITION 35 (Equality in Coherence Models). *Let $\langle \mathfrak{C}, \beta, v\rangle$, be a coherence model.*

$$\begin{array}{lll}
\langle \mathfrak{C}, \beta, v\rangle \vDash x \doteq y & :\Leftrightarrow & \tau(\beta(x), v) = \tau(\beta(y), v) \\
\langle \mathfrak{C}, \beta, v\rangle \vDash x \doteq^* y & :\Leftrightarrow & \textit{for all } w \in T(v) : \tau(\beta(x), w) = \tau(\beta(y), w) \\
\langle \mathfrak{C}, \beta, v\rangle \vDash x \approx y & :\Leftrightarrow & \textit{for all } w \in Z(v) : \tau(\beta(x), w) = \tau(\beta(y), w) \\
\langle \mathfrak{C}, \beta, v\rangle \vDash x \approx^+ y & :\Leftrightarrow & \textit{for all } w \in W : \tau(\beta(x), w) = \tau(\beta(y), w) \\
\langle \mathfrak{C}, \beta, v\rangle \vDash x \equiv y & :\Leftrightarrow & \beta(x) = \beta(y)
\end{array}$$

As it turns out, although all these notions are different semantically, we can only distinguish simple identity in trace from the other relations, that is to say, the latter four cannot be defined by means of modal axioms in the standard modal language using $\doteq$. Metatheoretically, the interrelations between $\approx$, $\approx^+$, $\doteq^*$ and $\equiv$ can—besides the usual axioms for identity (reflexivity, symmetry, transitivity)—be given by the following postulates.

$$\begin{array}{l}
\bigwedge x. \bigwedge y. x \equiv y \to x \approx^+ y \\
\bigwedge x. \bigwedge y. x \approx^+ y \to x \approx y \\
\bigwedge x. \bigwedge y. x \approx y \to x \doteq^* y \\
\bigwedge x. \bigwedge y. x \doteq^* y \to x \doteq y \\
\bigwedge x. \bigwedge y. x \doteq^* y \to \Box(x \doteq^* y)
\end{array}$$

Global identity in trace implies strong identity in trace if the frame is connected. A frame is called **cyclic** if for all v: $T(v) = Z(v)$. $\mathsf{S5}$–frames are cyclic. Tense frames are also cyclic. (Notice that we have not defined $T(v)$ for polymodal frames. See however [9] for a definition.) Inherent identity in trace implies global identity in cyclic frames. Finally, notice that it is always possible to factor out the equivalence relation $\equiv$. That is to say, without changing the logic we can identify all objects that have the same trace function. Effectively, a coherence frame in which $\equiv$ is the identity is the same as a worldline frame. This is the content of Proposition 17.

§9. Objects in Metaframes.

Shehtman and Skvortsov have introduced in [13] the metaframe semantics and shown that it is complete for all modal predicate logics. Their results were stated and proved for superintuitionistic logics and extensions of $\mathsf{S4}$. However, by removing some of the category theoretic definitions one can generalize these results easily to arbitrary modal predicate logic.

DEFINITION 36. *Σ denotes the category of finite ordinals and functions between them.*

DEFINITION 37 (Metaframes). *A* ***general metaframe*** *M is a contravariant functor from the category Σ into the category of general frames. In particular, for every n, $M(n) = \langle F_n, \lhd_n, \mathbb{F}_n\rangle$ is a general frame, and for each $\sigma : m \to n$, $M(\sigma)$ is a p–morphism from $M(n)$ to $M(m)$. A* ***metaframe*** *is a contravariant functor from Σ into the category of Kripke–frames. We call the members of F_n n–**points**.*

The idea is this. $M(0)$ represents the frame of possible worlds and $M(n)$ for $n > 0$ represents n–tuples over worlds. The arrows are needed to be able to identify the worlds and the tuples. For example, there is a unique map $0_n : 0 \to n$ for each n. Consequently, we have a map $M(0_n) : M(n) \to M(0)$. Thus, for each $a \in M(n)$, the **world** of a is $M(0_n)(a)$. Further, there is a (unique) natural embedding $i_{n,n+1} : n \to n+1 : i \mapsto i$. Hence, we define a **projection** of $a \in M(n+1)$ onto $M(n)$ by $M(i_{n,n+1})(a)$. We shall write $a \downarrow b$ if $a \in M(n+1)$ for some $n \in \omega$ and $b = M(i_{n,n+1})(a)$. Further, write $p^n_i : 1 \to n$ for the unique map sending 0 to i and if $\sigma : m \to n$, write $x^\sigma := \langle x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(m-1)}\rangle$.

DEFINITION 38 (Interpretations). *Let M be a general metaframe. An* ***interpretation*** *on M is a function ξ assigning to each predicate letter P an internal set of $M(\Omega(P))$, i.e. $\xi(P) \in \mathbb{F}_{\Omega(P)}$.*

For $a \in M(n)$ and $\sigma : m \to n$, $m = \Omega(P)$ we define

$$\begin{array}{lcl}
\langle M, \xi, a\rangle \vDash P(x^\sigma) & :\Leftrightarrow & M(\sigma)(a) \in \xi(P) \\
\langle M, \xi, a\rangle \vDash x_i \doteq x_j & :\Leftrightarrow & M(p_i^n)(a) = M(p_j^n)(a) \\
\langle M, \xi, a\rangle \vDash \neg\chi & :\Leftrightarrow & \langle M, \xi, a\rangle \nvDash \chi \\
\langle M, \xi, a\rangle \vDash \varphi \wedge \chi & :\Leftrightarrow & \langle M, \xi, a\rangle \vDash \varphi; \chi \\
\langle M, \xi, a\rangle \vDash \Diamond\chi & :\Leftrightarrow & \textit{exists } b \rhd_n a : \langle M, \xi, b\rangle \vDash \chi \\
\langle M, \xi, a\rangle \vDash \bigvee x_n.\chi & :\Leftrightarrow & \textit{exists } b \downarrow a : \langle M, \xi, b\rangle \vDash \chi
\end{array}$$

We can identify $M(0)$ with the sets of worlds, $M(1)$ with the sets of objects, $M(2)$ with the sets of pairs of objects, and so on. Now, the definitions will not suffice to define a MPL from a metaframe unless it satisfies a further condition. Let $\sigma : m \to n$. Write σ^+ for the unique function from $m+1$ to $n+1$ such that $\sigma^+ \restriction m = \sigma$ and $\sigma(m) = n$. Further, for $m \leq n$, $i_{m,n} : m \to n : j \mapsto j$ is the unique inclusion.

Definition 39 (Modal Metaframes). *A metaframe satisfies the* ***lift property*** *if for all $\sigma : m \to n$ and $a \in M(n)$, $b \in M(m+1)$ such that $M(\sigma)(a) = M(i_{m,m+1})(b) = d \in M(m)$ there exists a $c \in M(n+1)$ such that*

$$a = M(i_{n,n+1})(c) \text{ and } b = M(\sigma^+)(c)$$

A metaframe is a ***modal metaframe*** *if it satisfies the lift property.*

Shehtman and Skvortsov give a canonical procedure to obtain a modal metaframe from a modal predicate logic, see also Bauer [1]. Let L be given. We then let $M(n)$ be the set of all complete n–types $\Gamma = \Gamma(x_0, \dots, x_{n-1})$. They form a frame, where $\Gamma \lhd_n \Delta$ iff for all $\Box\varphi \in \Gamma$ we have $\varphi \in \Delta$. Further, if $\sigma : m \to n$, then $M(\sigma) : M(n) \to M(m)$ is defined by

$$M(\sigma)(\Delta) := \{\chi : [x_{\sigma(0)}/x_0, x_{\sigma(1)}/x_1, \dots, x_{\sigma(m-1)}/x_{m-1}]\chi \in \Delta\}$$

Actually, the definition of truth in a model can be changed somewhat. First, it can be shown that if the free variables of χ are in the set $\{x_i : i < n\}$, we have

$$\langle M, \xi, a\rangle \vDash \chi \quad \Leftrightarrow \quad \text{for all } b \downarrow a : \langle M, \xi, b\rangle \vDash \chi$$

In this form we see that the truth of a formula is upward persistent. (The abovementioned conditions on metaframes are such that this holds.)

Now write $a \sim_i c$ for the following. Let $\sigma : n \to n : i \mapsto n-1, n-1 \mapsto i, j \mapsto j$ ($j \notin \{i, n-1\}$). (If i is not less than $n-1$, this map is the identity.) Then $a \sim_i c$ iff there exist b such that $M(\sigma)(a) \downarrow b$ and $M(\sigma)(c) \downarrow b$. Then we have

$$\langle M, \xi, a\rangle \vDash \bigvee x_i.\chi \quad \Leftrightarrow \quad \text{there is } c \sim_i a : \langle M, \xi, c\rangle \vDash \chi$$

This is the form that we shall use later on. (It is closer in spirit to cylindrification.)

This construction is highly abstract. We shall demonstrate this construction with a very simple example. Our language contains only equality. Suppose that we have a logic L which contains $\varphi \leftrightarrow \Box\varphi$ for every formula φ. Then we have $\Gamma \lhd_n \Delta$ iff $\Gamma = \Delta$ for all n–types Γ and Δ. So, the relations are trivial. Suppose also that the logic contains the sentence saying that there are exactly three objects. We shall calculate the cardinalities of the $M(n)$. There is exactly one 0–type Γ_0, since the logic is POST–complete. There exists exactly one 1–type, since all objects are indistinguishable. There exist 2 2–types, namely the type containing $x_0 \doteq x_1$ and the other containing $x_0 \not\doteq x_1$. The general formula is as follows. Let n be given. Choose a function $f : \{0, 1, \dots, n-1\}$ into the set $\{0, 1, 2\}$. Then for this function the associated type is

$$t_f := \{x_i \doteq x_j : f(i) = f(j), i, j < n\} \cup \{\neg(x_i \doteq x_j) : f(i) \neq f(j), i, j < n\}$$

Now let $f \approx g$ iff there is a permutation $\pi : \{0, 1, 2\} \to \{0, 1, 2\}$ such that $f = \pi \circ g$. Obviously, $t_f = t_g$ iff $f \approx g$. (Case 1) $f(i) = f(j)$ for all $i, j < n$. There are 3 functions, all representing the same type. (Case 2) The image of f has at least two members. There are $3^n - 3$ many such functions. Each type is represented by six functions. This gives $(3^{n-1}-1)/2$ functions. In total we have $(3^{n-1}+1)/2$ functions. The series is

$$1, 2, 5, 14, 41, 122, \ldots$$

It is clear that the objects in such a frame are very difficult to recover. For this reason, Shehtman and Skvortsov define a *cartesian metaframe* (see below in Section 12 for a definition). This is a metaframe in which for each 0–type Γ the set of all n–types Δ with sentential reduct Γ is isomorphic to the n–fold cartesian product of the set of 1–types with sentential reduct Γ. Moreover, the projections are the maps $M(\iota_i^n)$, where $\iota_i^n : 1 \to n : 0 \mapsto i$. As Bauer [1] shows, each metaframe that allows to fuse types (a condition which we shall not define here) has a logically equivalent cartesian metaframe. The canonical metaframe defined above satisfies this condition. Thus, every logic is complete with respect to cartesian metaframes. Still, these proofs are very tedious. In the next section we shall show how the present results allow to prove completeness with respect to metaframe semantics using the previous completeness result.

§10. Going Second Order. In Kracht and Kutz [10] we have defined a notion of second order modal logics. Although they technically correspond to Π_1^1–formulae, we shall nevertheless call them second order logics. To be precise we shall describe them as logics over a slightly different language. Namely, while before we had a set Π of predicate symbols, now we assume to have predicate variables of any given arity.

Definition 40 (Symbols and Languages). *The languages of* ***second order modal predicate logic****, abbreviated collectively by* $\boldsymbol{MPL}^2$*, contain the following symbols.*

1. *A denumerable set* $V := \{x_i : i \in \omega\}$ *of* ***object variables****.*
2. *A denumerable set* $C := \{c_i : i \in \omega\}$ *of* ***constants****.*
3. *For each* $n \in \omega$*, a denumerable set* $PV^n := \{P_i^n : i \in \omega\}$ *of* ***predicate variables****.*
4. ***Boolean functors*** $\bot$, $\wedge$, $\neg$.
5. ***Quantifiers*** $\bigvee$, $\bigwedge$.
6. *A set* $M := \{\Box_\lambda : \lambda < \kappa\}$ *of* ***modal operators****.*

Furthermore, $\Omega(P_i^n) = n$ *for all* $n, i \in \omega$*.*

There may also be predicate constants but we have chosen to do without them for the purpose of simplicity, with the exception of equality. As before, we deal with only one modal operator. The generalization is obvious. The following substitution principle has first been discussed by Steven Kleene in his [8].

Definition 41 (Second Order Substitution). *Let* φ *and* χ *be formulae of* $\boldsymbol{MPL}^2$ *and* P *be a predicate variable. Then* $[\chi/P]\varphi$ *denotes the formula that is obtained by replacing every occurrence of* $[\vec{y}/\vec{x}]P$ *by* $[\vec{y}/\vec{x}]\chi$*, where bound variables get replaced in the usual (first–order) way to prevent accidental capture.*

We shall describe this substitution principle in a little more detail. Notice that χ can have free variables that are not among the variables $x_0, \ldots, x_{\Omega(P)-1}$. Let $\vec{z}$ be these variables. Then we replace φ by a bound variant φ', in which all variables of φ occurring in $\vec{z}$ are replaced by suitable variables not occurring in either φ or χ. Next, we perform the replacement of any occurrence of $[\vec{y}/\vec{x}]P$ for some variables $\vec{y}$ by $[\vec{y}/\vec{x}]\chi$. This time, no bound variant needs to be chosen. For example, let $\varphi = \bigvee x_2. \bigwedge x_0. \bigwedge x_1.P_0^1(x_2) \wedge P_1^2(x_0, x_2) \to P_1^2(x_1, x_0)$. Let $\chi = \bigvee x_1.P_0^3(x_2, x_1, x_0)$. Suppose that we want to replace P_1^2 by χ. Then since x_2 occurs free in χ, we shall replace bound occurrences of x_2 in φ by x_4. This gives

$$\varphi' = \bigvee x_4. \bigwedge x_0. \bigwedge x_1.P_0^1(x_2) \wedge P_1^2(x_0, x_4) \to P_1^2(x_1, x_0)$$

Finally, we have to replace $P_1^2(x_0, x_4)$ by $[x_4/x_1]\chi = \bigvee x_1.P_0^3(x_2, x_1, x_0)$ and also $P_1^2(x_1, x_0)$ by $[x_1/x_0, x_0/x_1]\chi = \bigvee x_3.P_0^3(x_2, x_3, x_1)$.

$$[\chi/P_1^2]\varphi = \bigvee x_4. \bigwedge x_0. \bigwedge x_1.P_0^1(x_2) \wedge \bigvee x_1.P_0^3(x_2, x_1, x_0) \to \bigvee x_3.P_0^3(x_2, x_3, x_1)$$

This language does not contain an existence predicate constant, but this is only for convenience. There are no further complications in introducing predicate constants as well, but we have omitted them here to keep the notation reasonably simple. The following definition is analogous to Skvortsov and Shehtman [13].

Definition 42 (Second Order MPLs). *A **second order MPL** is a set L of $\boldsymbol{MPL}^2$–formulae satisfying the following conditions.*

1. *L contains all instances of axioms of first–order logic.*
2. *L is closed under all rules of first–order logic.*
3. *L is closed under second–order substitution.*
4. *L contains all instances of axioms of the modal logic* K.
5. *L is closed under the rule $\varphi/\Box\varphi$.*
6. $\Diamond \bigvee y.\varphi \leftrightarrow \bigvee y.\Diamond\varphi \in L$.

Clearly, a second order MPL can be viewed as a special sort of a first–order MPL, by taking Π to be the union of the sets PV^m. This allows us to speak, for example, of the canonical structure $\mathfrak{Can}_{L^*}$ for L. However, these languages are technically distinct, since the predicate variables are not interpreted in the structure. Their value is not fixed in the structure, just like the value of an object variable is not fixed in a first–order structure. This means that technically we get a different notion of structure. However, the way we get these structures is by abstracting them from the corresponding first–order structures. Thus, we begin with a second order MPL L and interpret it as a first–order MPL, also called L, for which we then build the canonical structure $\mathfrak{Can}_{L^*}$. Starting with this structure we shall define the second–order structure for L^*.

Definition 43 (Second Order Coherence Frames). *A **second order coherence frame** is a triple $\langle W, \lhd, U\rangle$, where $\langle W, \lhd\rangle$ is a Kripke–frame and U a set.*

Given a second order coherence frame, we call a member of $W \times U^n$ an n–**point** and a subset of $W \times U^n$ an n–**set** . Let $p = \langle v, \vec{a}\rangle$ and $q = \langle w, \vec{c}\rangle$ be n–points. We write $p \sim_k q$ if $a_i = c_i$ for all $i \neq k$.

$$\mathsf{V}_k(A) := \{q : \text{exists } p \in A : p \sim_k q\}$$

(If $k < n$ does not obtain, then we may put $\mathsf{V}_k(A) := A$.) Next, let $\sigma : m \to n$. Then we define $\widehat{\sigma}$ on n–points as follows.

$$\widehat{\sigma}(\langle v, \vec{a}\rangle) := \langle v, \langle a(\sigma(i)) : i < m\rangle\rangle$$

If p is an n–point, $\widehat{\sigma}(p)$ is an m–point. So, $\widehat{-}$ is a contravariant functor from Σ into the set of functions from points to points. This is also directly verified. If $\tau : \ell \to m$ then

$$\begin{array}{rcl} \widehat{\sigma \circ \tau}(p) & = & \langle v, \langle a(\sigma \circ \tau(i)) : i < \ell\rangle\rangle \\ & = & \widehat{\sigma}(\langle v, \langle a(\tau(i)) : i < \ell\rangle\rangle) \\ & = & \widehat{\tau}(\widehat{\sigma}(p)) \\ & = & \widehat{\tau} \circ \widehat{\sigma}(p) \end{array}$$

For an m–set A we put

$$\mathsf{C}_\sigma(A) := \{p : \widehat{\sigma}(p) \in A\}$$

It is directly verified that C is covariant, that is, $\mathsf{C}_{\sigma\circ\tau} = \mathsf{C}_\sigma \circ \mathsf{C}_\tau$. For we have for an ℓ–set A:

$$\begin{array}{rcl} \mathsf{C}_{\sigma\circ\tau}(A) & = & \{p : \widehat{\sigma \circ \tau}(p) \in A\} \\ & = & \{p : \widehat{\tau}(\widehat{\sigma}(p)) \in A\} \\ & = & \{p : \widehat{\sigma}(p) \in \mathsf{C}_\tau(A)\} \\ & = & \mathsf{C}_\sigma(\mathsf{C}_\tau(A)) \\ & = & \mathsf{C}_\sigma \circ \mathsf{C}_\tau(A) \end{array}$$

And finally we set

$$\blacklozenge A := \{\langle w, \vec{a}\rangle : \text{exists } v \rhd w : \langle v, \vec{a}\rangle \in A\}$$

DEFINITION 44 (n–Complexes and Towers). *An n–**complex** over a 2nd order coherence frame is a set $\mathbb{C}_n$ of n–sets closed under intersection, complement, the operations V_k, $\blacklozenge$ and C_σ for every $\sigma : n \to n$. A **tower** is a sequence $\langle \mathbb{C}_n : n \in \omega \rangle$ such that $\mathbb{C}_n$ is an n–complex for every n, and for every $\sigma : n \to m$, $\mathsf{C}_\sigma : \mathbb{C}_n \to \mathbb{C}_m$.*

DEFINITION 45 (Generalized Second Order Coherence Frames). *A **generalized second order coherence frame** is a quadruple $\mathfrak{S} = \langle W, \lhd, U, \mathfrak{T} \rangle$, where $\langle W, \lhd, U \rangle$ is a second order coherence frame and $\mathfrak{T} = \langle \mathbb{C}_i : i \in \omega \rangle$ a tower over it. A **valuation** into $\mathfrak{S}$ is a pair ξ and β of mappings, where ξ is defined on all predicate variables, and $\xi(P_i^m) \in \mathbb{C}_m$ for all $m, i \in \omega$ and β assigns to each $x_i \in V$ a member of U.*

$$\begin{array}{lll}
\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash P_i^m(\vec{y}) & :\Leftrightarrow & \langle \beta(y_i) : i < m \rangle \in \xi(P_i^m) \\
\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \neg\chi & :\Leftrightarrow & \langle \mathfrak{S}, \xi, \beta, v \rangle \nvDash \chi \\
\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \varphi \wedge \chi & :\Leftrightarrow & \langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \varphi; \chi \\
\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \bigvee y.\chi & :\Leftrightarrow & \textit{for some } \beta' \sim_y \beta : \langle \mathfrak{S}, \xi, \beta', v \rangle \vDash \chi \\
\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \Diamond\chi & :\Leftrightarrow & \textit{for some } w \rhd v : \langle \mathfrak{S}, \xi, \beta, w \rangle \vDash \chi
\end{array}$$

We write $\mathfrak{S} \vDash \varphi$ if for all valuations ξ, β and all worlds v: $\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \varphi$.

Notice that Shehtman and Skvortsov define the truth of a formula at an n–point. We can do the same here. Namely, we set

$$\langle \mathfrak{S}, \xi, \langle v, \vec{a} \rangle \rangle \vDash \varphi$$

iff for any valuation β such that $\beta(x_i) = a_i$ for all $i < n$ we have

$$\langle \mathfrak{S}, \xi, \beta, v \rangle \vDash \varphi$$

An inductive definition can be given as well. We can also define a valuation on a metaframe in the following way. We say that a function $\beta : V \to \bigcup_n M(n)$ is a **valuation** if $\beta(x_n) \in M(n)$, and if $\beta(x_{n+1}) \downarrow \beta(x_n)$ for every n. Thus, $\beta(x_n)$ is an n–point which is the projection of the $n+1$–point $\beta(x_{n+1})$.

We shall show that second order MPLs are complete with respect to this semantics. So, let L be a second order MPL. We understand it as a first–order MPL, which we denote by the same letter. Then we can construct the canonical first–order coherence structure $\mathfrak{Can}_{L^*} = \langle W_{L^*}, \lhd, C_{L^*}, \mathfrak{I}_{L^*} \rangle$. We shall now define a second order canonical *frame* from it. This construction is completely general. First, observe that we can transport the notion of n–point as well as the satisfaction of a formula at an n–point to frames (and first–order coherence frames).

Take a modal (first–order) structure $\mathfrak{S} = \langle W, \lhd, U, \mathfrak{I} \rangle$. Let φ be a formula such that the free variables occurring in it are contained in $\{x_i : i < n\}$. Then we write $[\varphi]_n$ for the set of n–points satisfying φ. Formally,

$$[\varphi]_n := \{p : \langle \mathfrak{S}, p \rangle \vDash \varphi\}$$

Now set

$$\mathbb{C}_n := \{[\varphi]_n : \mathit{fvar}(\varphi) \subseteq \{x_i : i < n\}\}$$

Finally, we put $\mathfrak{T} := \langle \mathbb{C}_n : n \in \omega \rangle$. Now let

$$\mathfrak{S}^2 := \langle W, \lhd, U, \mathfrak{T} \rangle$$

LEMMA 46. *$\mathfrak{S}^2$ is a second order generalized coherence frame. Furthermore, with $\xi(P) := \mathfrak{I}(P)$ we have for every n–point p and every formula φ with free variables in $\{x_i : i < n\}$:*

$$\langle \mathfrak{S}, p \rangle \vDash \varphi \quad \Leftrightarrow \quad \langle \mathfrak{S}^2, \xi, p \rangle \vDash \varphi$$

Proof. We need to verify that $\mathfrak{T}$ is a tower. This follows from the following equations. (In the last clause, $\sigma : n \to m$.)

$$\begin{array}{lll}
[\neg\chi]_n & = & -[\chi]_n \\
[\varphi \wedge \chi]_n & = & [\varphi]_n \cap [\chi]_n \\
[\bigvee x_k.\chi]_n & = & \mathsf{V}_k([\chi]_n) \\
[\Diamond\chi]_n & = & \blacklozenge[\chi]_n \\
[[\vec{x}^\sigma/\vec{x}]\chi]_n & = & \mathsf{C}_\sigma([\chi]_m)
\end{array}$$

Only the last clause needs comment.

$$\begin{aligned}[[\vec{x}^\sigma/\vec{x}]\chi]_n &= \{p \in U \times W^n : \langle \mathfrak{S}, p\rangle \vDash [\vec{x}^\sigma/\vec{x}]\chi\} \\ &= \{p \in U \times W^n : \langle \mathfrak{S}, \widehat{\sigma}(p)\rangle \vDash \chi\} \\ &= \{p \in U \times W^n : \widehat{\sigma}(p) \in [\chi]_m\} \\ &= \mathsf{C}_\sigma([\chi]_m)\end{aligned}$$

The second claim is immediate to verify. ⊣

THEOREM 47 (Second Order Completeness). *Let L be a second order modal logic without equality with the canonical structure $\mathfrak{Can}_{L^*}$ and φ a formula. Then $\mathfrak{Can}^2_{L^*} \vDash \varphi$ iff $\varphi \in L$. It follows that L is complete with respect to second–order generalized coherence frames.*

Proof. We pass to the first–order canonical structure $\mathfrak{Can}_{L^*}$ of the (first–order) MPL L. Let ξ_{L^*} be defined by

$$\xi_{L^*}(P_i^m) := [P_i^m(x_0, \dots, x_{m-1})]_m$$

Then, by first order completeness and Lemma 46 we get

$$\langle \mathfrak{Can}^2_{L^*}, \xi_{L^*}\rangle \vDash \varphi \Leftrightarrow \varphi \in L$$

We have to show that if $\varphi \in L$ then for every valuation ξ into $\mathfrak{Can}^2_{L^*}$ we have

$$\langle \mathfrak{Can}^2_{L^*}, \xi\rangle \vDash \varphi$$

For this establishes $\mathfrak{Can}^2_{L^*} \vDash \varphi$ in case $\varphi \in L$. If $\varphi \notin L$ then we have anyway

$$\langle \mathfrak{Can}^2_{L^*}, \xi_{L^*}\rangle \nvDash \varphi$$

by first–order completeness and Lemma 46. Now for the proof of the claimed fact. Assume that ξ is given. By definition of the tower $\mathfrak{T}_{L^*}$ there exists for every predicate variable P a formula $\pi_P(\vec{x})$ such that

$$\xi(P) = \xi_{L^*}(\pi_P(\vec{x}))$$

Let $pvar(\varphi)$ denote the set of predicate variables occurring in φ. Define

$$\varphi_* := [\pi_P/P : P \in pvar(\varphi)]\varphi$$

This formula is unique up to renaming of bound variables. Then, by induction, it is verified that

$$\langle \mathfrak{Can}^2_{L^*}, \xi, \Delta\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle \mathfrak{Can}^2_{L^*}, \xi_{L^*}, \Delta\rangle \vDash \varphi_*$$

Since $\varphi \in L$ and L is closed under second order substitution we have $\varphi_* \in L$ as well. Hence the right hand side obtains, and therefore the left hand side is true as well. This is what we had to prove. ⊣

This construction of retracting the valuation ξ and adding the tower of definable sets is applicable to any first order coherence structure.

§11. A Logic That is Incomplete with Respect to Coherence Frames. In this section we will give an axiomatization of a 2nd order modal predicate logic that is the logic of a single counterpart frame having two distinct counterpart relations. This logic will also turn out not to be valid on any coherence frame. Define the following counterpart frame $\mathfrak{F}$: let $W = \{w\}$ be the set of worlds, $U_w = \{a, b\}$ be the universe of w and $\mathcal{C}(w, w) = \{f, g\}$ the set of counterpart relations from w to itself, where $f : a \to a, \quad b \to b$ and $g : a \to b, \quad b \to a$. Notice that this frame is not object rich, for $\langle a, b\rangle$ is a thread in $\mathfrak{F}$ but there is no object that leaves both a and b as its trace in w.

Call Λ the second order MPL axiomatized as follows:

(a) B: $p \to \Box\Diamond p$, T: $p \to \Diamond p$, $\mathsf{4}$: $\Diamond\Diamond p \to \Diamond p$
(b) alt_2: $\Diamond p \wedge \Diamond q \wedge \Diamond r \to \Diamond(p \wedge q) \vee \Diamond(p \wedge r) \vee \Diamond(q \wedge r)$
(c) $\bigwedge x_0, x_1, x_2.(x_0 \doteq x_1 \vee x_1 \doteq x_2 \vee x_0 \doteq x_2)$
(d) $\bigvee x_0, x_1.\neg(x_0 \doteq x_1)$
(e) $\bigwedge x_0, x_1.x_0 \doteq x_1 \to \Box(x_0 \doteq x_1)$

(f) $\bigwedge x_0, x_1.\neg(x_0 \doteq x_1) \to \Box\neg(x_0 \doteq x_1)$
(g) $\bigwedge x_0, x_1.P(x_0) \wedge \neg(x_0 \doteq x_1). \to .\Box(P(x_0) \vee P(x_1))$

Clearly, $\mathfrak{F} \vDash \Lambda$. In counterpart frames, the axioms (c) and (d) together state that there exist exactly 2 things in each world. When interpreted in coherence frames, they state that there are exactly two object traces in each world, but there may still be infinitely many objects. However, by equivalentiality, this implies that at most two objects are discriminable in each world.

Furthermore, (e) and (f) state that identity and difference are necessary. Finally, (g) states that whatever applies to a given object either applies to it in a successor world, or to the other object. It may be arguable from a philosophical point of view whether Λ is a genuine logic, because it makes *existence assumptions* about e.g. the number of objects or object traces in the world. Nevertheless, it surely is a logic in the technical sense of Definition 2.

THEOREM 48. *There is no class $\mathcal{K}$ of Lewisian counterpart frames such that Λ is the second order modal logic of $\mathcal{K}$. There is no class $\mathcal{C}$ of coherence frames such that Λ is the second order modal logic of $\mathcal{C}$.*

Proof. The propositional reduct of this logic is the logic $\mathsf{S5.alt}_2$, which possesses exactly two nonisomorphic Kripke frames, namely the 1–point reflexive frame and the 2–point frame, where accessibility is universal. Furthermore, by necessity of identity and distinctness and the fact that the number of objects is constant and finite, counterpart relations are bijective functions. Suppose now that we have two worlds, v and w. Suppose further that there is a formula φ that is true of a but not of b in v, where a and b are the objects of the domain of v. Then, by virtue of (g), φ can be true only of one of the objects in w. There is however nothing that guarantees this if $v \neq w$. This argument is valid both for counterpart frames and coherence frames. So, we can have only one world. It remains to show that if there is only one world then the logic of the frame is stronger than Λ. Now, if $\mathfrak{C}$ is a coherence frame containing only one self–accessible world or if it is a counterpart frame containing only one world with one counterpart relation, then $\mathfrak{C} \vDash \Diamond p \to \Box p$, which is not a theorem of Λ, since $\mathfrak{F} \nvDash \Diamond p \to \Box p$. ⊣

It follows that whereas Λ is characterized by its canonical coherence model, compare Theorem 15, there is no coherence frame in which Λ is valid, that is to say, Λ is coherence frame incomplete. Evidently, if we add the right towers, completeness is regained (of course, with respect to generalized frames). On the other hand, we have shown in Section 7 how the notion of coherence frame can be modified to gain the same expressive power as counterpart semantics without moving to the full second–order semantics.

§12. Cartesian Metaframes. In this section we shall use the previous completeness proof to derive a very simple completeness proof for the metaframe semantics. Shehtman and Skvortsov give the following definition.

DEFINITION 49 (Cartesian Metaframes). *A metaframe M is called **cartesian** if the following holds.*

1. *There is a set U and a family $\{W_u : u \in U\}$ of nonempty and pairwise disjoint sets such that $M(0) = U$ and $M(n) = \bigcup_{u \in U}(W_u)^n$ for every n. We write $a \lhd_1 b$ iff there are $u, v \in U$ such that $a \in W_u$, $b \in W_v$, $u \lhd_0 y$ and $\langle u, a\rangle \lhd_1 \langle v, b\rangle$.*
2. *$\langle v, \vec{a}\rangle \lhd_n \langle w, \vec{c}\rangle$ iff*
   (a) *$v \lhd_0 w$*
   (b) *$a_i \lhd_1 c_i$ for all $i < n$ and*
   (c) *for all $i < j < n$: if $a_i = a_j$ then also $c_i = c_j$.*
3. *For every $\sigma : n \to m$, $m, n > 0$, $M(\sigma) = \widehat{\sigma} : \vec{a} \mapsto \langle a(\sigma(i)) : i < n\rangle$. For $\sigma : 0 \to n$, $n > 0$, $M(\sigma) : \vec{a} \mapsto u$, where u is such that $\vec{a} \in (W_u)^n$.*

For $\sigma : n \to 0$, there is no definition of $M(\sigma)$ given. One possibility is to choose an object $u^* \in W_u$ for every $u \in U$ and then let $M(\sigma) : u \mapsto (u^*)^n$. It is an approximation of the idea that the elements of the nth frame are n–tuples. While cartesian metaframes assume that the n–tuples are tuples of things, we shall offer another variant, where the idea is that the tuples are in fact tuples of objects.

DEFINITION 50 (Cubic Metaframes). *A metaframe M is called **cubic** if the following holds.*

1. *There are sets U and W such that $M(0) = U$ and $M(n) = U \times M^n$ for every n.*
2. *$\langle u, \vec{a}\rangle \lhd_n \langle v, \vec{c}\rangle$ iff $\vec{c} = \vec{a}$ and $u \lhd_0 v$.*
3. *For every $\sigma : n \to m$, $M(\sigma) = \widehat{\sigma} : \langle u, \vec{a}\rangle \mapsto \langle u, \langle a(\sigma(i)) : i < n\rangle\rangle$.*

It is first of all to be checked that the above requirements define a contravariant functor from Σ to the class of generalized frames. (a) $M(n)$ is a general frame, as is easily seen. (b) for each $\sigma : m \to n$, $M(\sigma)$ is a p–morphism from $M(n)$ to $M(m)$. Namely, suppose that $p = \langle v, \vec{a}\rangle \lhd_n \langle w, \vec{c}\rangle = q$. Then $\vec{c} = \vec{a}$ and $v \lhd w$. Hence $\langle a(\sigma(i)) : i < m\rangle = \langle c(\sigma(i)) : i < m\rangle$, and so $\widehat{\sigma}(p) \lhd_m \widehat{\sigma}(q)$. Second, suppose that $\widehat{\sigma}(p) \lhd_m q'$. Then $q' = \langle w', c'(i) : i < m\rangle$ for some w' such that $v \lhd w'$ and $c'(i) = a(\sigma(i))$ for each $i < m$. So, put $q := \langle w', \vec{a}\rangle$. Then $p \lhd_n q$ and $M(\sigma)(q') = q$. Third, let $A \in \mathfrak{T}(m)$. Then $\mathsf{C}_\sigma(A) \in \mathfrak{T}(n)$, by definition of towers. This proves that $M(\sigma)$ is a p–morphism. (c) For each $\sigma : m \to n$ and $\tau : n \to q$, $M(\tau \circ \sigma) = M(\sigma) \circ M(\tau)$. But by previous calculations, $M(\tau \circ \sigma)(p) = \widehat{\sigma} \circ \widehat{\tau}(p) = M(\sigma) \circ M(\tau)(p)$.

PROPOSITION 51. *For every cubic metaframe M there exists a semantically equivalent cartesian metaframe N.*

Proof. Let M be a cubic metaframe. Put $N(0) := M(0)$ and $W_u := \{u\} \times W$ for all $u \in U$ and $\langle u, a\rangle \lhd_1 \langle v, b\rangle$ iff $u \lhd_0 v$ and $a = b$. Then $N(n) := \{\langle\langle u, a_i\rangle : i < n\rangle : a_i \in W\}$ for all n and $\vec{a} \lhd_n \vec{c}$ iff there are u and v such that $u \lhd_0 v$ and $a_i = \langle u, o_i\rangle$, $c_i = \langle v, o_i\rangle$ for some $o \in W$. The p–morphisms $M(\sigma)$ are straightforwardly defined. ⊣

The reason that this works is a construction that we have used before: the trace of an object at u may be the pair consisting of u and the object itself. It is easy to establish a bijective correspondence between second order generalized coherence frames and cubic generalized metaframes. Given a second order coherence frame $\langle W, \lhd, U, \mathfrak{T}\rangle$, we simply define $M(n) := \langle W \times U^n, \lhd_n, \mathfrak{T}(n)\rangle$, where $\langle v, \vec{a}\rangle \lhd_n \langle w, \vec{c}\rangle$ iff $\vec{a} = \vec{c}$ and $v \lhd w$. This is a cubic generalized metaframe. Conversely, let M be a cubic generalized metaframe, with $M(n) = \langle W \times U^n, \lhd_n, \mathbb{M}_n\rangle$ for every n. Then let $\lhd := \lhd_0$ and $\mathfrak{T}(n) := \mathbb{M}_n$. Then it is easily checked that $\langle W, \lhd, U, \mathfrak{T}\rangle$ is a second order general coherence frame. As an immediate consequence we get the following

THEOREM 52. *Every second order MPL without equality is complete with respect to cubic generalized metaframes.*

Notice that the way this result has been obtained is by abstraction from the first–order case, rather than the first–order case being an application of the second–order case. Completeness with respect to cartesian generalized metaframes now follows, given that cubic metaframes are special cartesian metaframes.

§13. Equality. Let us turn to the treatment of equality in coherence frames and metaframes. We have argued earlier that two objects may be equal in one world and different in another. Equality in a world has been regulated by the trace function. The most direct way to account for equality between objects is therefore to add the trace function into the generalized coherence frame. Another way is to add equality as a predicate constant whose interpretation is a equivalence relation on U in each individual world. Thus, we add a constant $\Delta \in \mathbb{C}_2$ such that the following holds. Write $a \; \Delta_w \; b$ iff $\langle w, a, b\rangle \in \Delta$.

1. $a_0 \; \Delta_w \; a_0$ for all $w \in W$, $a_0 \in U$.
2. If $a_0 \; \Delta_w \; a_1$ then $a_1 \; \Delta_w \; a_0$, for all $w \in W$, $a_0, a_1 \in U$.
3. If $a_0 \; \Delta_w \; a_1$ and $a_1 \; \Delta_w \; a_2$ then $a_0 \; \Delta_w \; a_2$ for all $w \in W$, $a_0, a_1, a_2 \in U$.

Additionally, a valuation must satisfy the following property. Call $A \in \mathbb{C}^n$ **equivalential** if for all $p = \langle w, \vec{a}\rangle \in A$, $q = \langle w, \vec{b}\rangle$ such that $a_i \; \Delta_w \; b_i$ for all $i < n$ then $q \in A$. Then we require that for every predicate P, $\xi(P)$ must be an equivalential set. However, notice that equivalential sets are not closed under $\blacklozenge$!

It may be disappointing to see that we have not been able to reduce $\doteq$ to simple identity. However, there is to our knowledge no semantics under which this is so (and for reasons given below it is not to be expected either). Let us look for example at metaframes. In a metaframe, $\langle M, \xi, a\rangle \vDash x_i \doteq x_j$ if $pr_{n,i}(a) = pr_{n,j}(a)$. Furthermore, a frame interpreting $\doteq$ (called an $m^=$–**metaframe**) must satisfy the following requirement:

($0^\sharp$) For all n and $a, b \in M(n)$ $a = b$ iff $pr_{n,i}(a) = pr_{n,i}(b)$.

This condition effectively eliminates the distinction between object and trace. However, in general metaframes the possibility of distinct developments for identical objects still exists: let a be in $M(2)$. Think of a as the pair $\langle a_0, a_1\rangle$. If $pr_{2,0}(a) = pr_{2,1}(a)$, then $a_0 = a_1$. Now, accessibility is a relation between pairs, so if $a \lhd_2 b = \langle b_0, b_1\rangle$, we may or may not have $b_0 = b_1$. If we move to cartesian metaframes, the situation is different, however. For now, if $a_0 \lhd_1 b_0$ and $a_1 \lhd_1 b_1$ then from $a_0 = a_1$ we expect $a \lhd_2 \langle b_0, b_0\rangle, \langle b_1, b_1\rangle, \langle b_1, b_0\rangle$ as well. Shehtman and Skvortsov make some maneuvers to avoid this consequence.

First, let us look at a definition of cartesian metaframes and assume that the clause that $a_i = a_j$ implies $c_i = c_j$ was not there. Then the following principle is valid. If $\varphi(y, \vec{z})$ is a formula such that x_0 and x_1 do not occur in $\vec{z}$ then

$$M \vDash \bigwedge x_0.\bigwedge x_1.x_0 \doteq x_1 \to (\Diamond[x_0/y]\varphi(y,\vec{z}) \leftrightarrow \Diamond[x_1/y]\varphi(y,\vec{z}))$$

For let ξ be a valuation and $a \in M(n)$. Assume that $\langle M, \xi, a\rangle \vDash x_0 \doteq x_1$. Then we have $pr_{n,0}(a) = pr_{n,1}(a)$. Let $\alpha_0 : n-1 \to n : i \mapsto i+1$ and $\alpha_1 : n-1 \to n : 0 \mapsto 0, i \mapsto i+1 (i \neq 0)$. Put $a_0 := M(\alpha_0)(a)$, $a_1 := M(\alpha_1)(a)$. Intuitively, a_0 is a reduced by its 0th coordinate, a_1 is a reduced by its 1st coordinate. From this it follows that $a_0 = a_1$. Assume next that $\langle M, \xi, a\rangle \vDash \Diamond[x_0/y]\varphi(y,\vec{z})$. Then there is a b such that $a \lhd_n b$ and $\langle M, \xi, b\rangle \vDash [x_0/y]\varphi(y,\vec{z})$. Put $b_0 := M(\alpha_0)(b)$ and $b_1 := M(\alpha_1)(b)$. We have $a_0 \lhd_{n-1} b_0$, since $M(\alpha_0)$ is a p–morphism, and likewise $a_1 \lhd_{n-1} b_1$. Since x_0 is not free in $[x_1/y]\varphi(y,\vec{z})$ we have $\langle M, \xi, b_1\rangle \vDash [x_0/y]\varphi(y,\vec{z})$. From this follows $\langle M, \xi, a_1\rangle \vDash \Diamond[x_0/y]\varphi(y,\vec{z})$, and so $\langle M, \xi, a\rangle \vDash \Diamond[x_1/y]\varphi(y,\vec{z})$. Likewise for the other direction.

So we find, as indicated, that without the clause, metaframes imitate counterpart semantics. However, Shehtman and Skvortsov have added it. Thereby they avoid counterpart semantics, but there is a price to be paid.

Lemma 53. *Let M be a cartesian $m^=$–metaframe. Then*

$$M \vDash \bigwedge x_0.\bigwedge x_1.x_0 \doteq x_1 \to \Box(x_0 \doteq x_1).$$

So, neither of the alternatives is completely general. It turns out that metaframe semantics could have been saved in the same way as coherence semantics, namely by adding a constant interpreting equality. This seems to be necessary. If we do not treat equality in this way, we must assume that the interpretation of identity is an equivalence relation. Shehtman and Skvortsov have shown that the condition ($0^\sharp$) makes the semantics less general: there are formulae which are not generally valid but valid in all metaframes satisfying ($0^\sharp$). This indicates that equating objects and object traces even done in metaframes à la Shehtman and Skvortsov cannot eliminate the problems of identity.

§14. Conclusion. In this paper we have defined a new semantics for modal predicate logic, namely coherence frames. Coherence frames differ from counterpart frames in that variables are interpreted in the same way as constants, namely by objects. We have shown completeness both for first–order and for second–order MPLs with respect to generalized coherence frames. From this we have derived a completeness theorem for second order MPLs with respect to generalized metaframe semantics. In fact, completeness with respect to cubic generalized metaframes is obtained rather directly.

The proposal of distinguishing between an object and its trace is certainly a very far reaching one but not without justification. Many philosophers have argued that there may exist different identity criteria for objects (see van Leeuwen [15] for a review of these ideas). A statue is not the same as the material it is made of. Hence, though perhaps trace identical, the two are not the same objects. There are predicates that are sensitive to this difference (again see [15]). These predicates reject the postulate Eq4. It goes beyond the scope of this paper to review the possibilities that coherence structures offer in this respect.

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