Lattices of Modal Logics and their Groups of Automorphisms

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Abstract. The present paper investigates the groups of automorphisms for some lattices of modal logics. The main results are the following. The lattice of normal extensions of **S4.3**, NExt **S4.3**, has exactly two automorphisms, NExt $\mathbf{K.alt_1}$ has continuously many automorphisms. Moreover, any automorphism of NExt **S4** fixes all logics of finite codimension. We also obtain the following characterization of pretabular logics containing **S4**: a logic properly extends a pretabular logic of NExt **S4** iff its lattice of extensions is finite and linear.

1. Introduction

Depending on circumstances, one may define a logic to be either a set of inference rules or a set of tautologies. These notions are clearly distinct; two different sets of inference rules may give rise to the same set of tautologies. A third notion, that is in between the two, is the notion of a consequence relation. The same consequence relation can be axiomatized differently by means of rules, and a given set of tautologies can be the set of tautologies of different consequence relations. Each of these three notions is significant in its own right. A definition of a logic by a set of inference rules takes a logic to talk about proofs, a definition by a consequence relation takes logics to talk about consequence and the definition of a logic by a set of tautologies takes a logic to talk about truth. In modal logic the situation is somewhat simplified by the fact that the set of proper rules is usually fixed (it contains only Modus Ponens). Hence, the consequence relation is no more informative than the set of tautologies.

There is an even more abstract way of studying logics, namely by their lattice of theories or their lattice of extensions. Also this has its motivation, namely focussing on the notion of *expressivity*. If we study, for example, the lattice of theories of a logic, we ask: In what ways can the formulae of the language discriminate states–of–affairs? To give an easy example: If there are κ many theories, only κ many states–of–affairs can be discriminated. If we study the lattice of axiomatic extensions we ask: in what ways can the formulae of the language discriminate logics? Moreover, it would be interesting to study to what extent a logic is determined by its lattice of extensions. Although we will not directly deal with this problem, some answers will be obtained in this paper as well.

In this paper we study the groups of automorphisms of lattices $\mathrm{NExt}\,\Lambda$ of normal extensions of certain modal logics Λ. This question makes of course sense independently of any motivation and has a similar significance as the studying the automorphisms of the lattice of Turing degrees. However, in trying to establish the structure of these groups we often meet the following problem, which we think is of independent interest. Namely, you are given a logic Θ and some lattice $\mathfrak{L}$ of extensions of a logic Λ, $\Lambda \subseteq \Theta$, together with an element x of $\mathfrak{L}$. Can you say whether x is the logic Θ? The answer to this question depends on the way in which the objects are given. If Θ is given as a set of tautologies, and $\mathfrak{L}$ simply *is* the same as $\mathrm{NExt}\,\Lambda$, then x is Θ iff $x = \Theta$. (If Θ is given by means of an axiomatization, the answer may however also depend on the decidability of Θ.) However, we want to analyse the situation that we are given $\mathrm{NExt}\,\Lambda$ only up to isomorphism. In that case, the question should be modified slightly to account for the fact that $\mathfrak{L}$ can in many ways be mapped onto $\mathrm{NExt}\,\Lambda$. So, the question is therefore the following:

> Let Λ and Θ be normal logic and $\Theta \supseteq \Lambda$. Given $x \in \mathfrak{L}$, is $i(x) = \Theta$ for all isomorphisms $i : \mathfrak{L} \to \mathrm{NExt}\,\Lambda$?

The answer is rather easy. It is positive if (1) Θ is fixed under all automorphisms of $\mathrm{NExt}\,\Lambda$, and (2) there is *some* isomorphism $i : \mathfrak{L} \to \mathrm{NExt}\,\Lambda$ such that $i(x) = \Theta$. Hence, we can make the question independent of $\mathfrak{L}$ and ask simply: Is an element fixed by all automorphisms? Now assume that Θ is fixed under all automorphisms. Still, determining whether or not whether for a given element $x \in \mathfrak{L}$ $i(x) = \Theta$ is far from trivial. For example, we do not know of any criterion that would allow us to identify **S4** or **K4** in the lattice of normal modal logics. (We do however also not know whether they are fixed under all automorphisms.) The lattice of normal extension of **K** is so complex that we have at present no hope of being able to attack this problem. In the present paper we make the simplifying assumption that Θ is the logic of a finite, rooted frame $\mathfrak{F}$. This frame is then unique up to isomorphism. In that case Θ is a strictly meet–irreducible logic and has finite codimension. Since every logic of finite codimension is the intersection of strictly meet–irreducible logics (which can be effectively computed either on the basis of $\mathfrak{F}$ or on the basis of x), the first is no restriction in view of the second. Let us however note that it makes a difference whether Θ is given by means of an axiomatization or by means of a finite frame. For in general it is undecidable given a finite frame $\mathfrak{F}$ and a finite set X of axioms whether or not $\mathbf{K} \oplus X$ is the theory of $\mathfrak{F}$. (This has been shown by A. Chagrov. See [6] for a proof.) Now, if a tabular logic contains **K4** this question becomes in fact decidable. For a tabular logic has a representation the form $\mathbf{K4}/N$, where N is a finite set of finite frames. We will with the exception of §8 assume that our logics contain **K4**. If that is assumed, we have reduced the problem to the following question.

> Let Λ be a normal modal logic with $\mathrm{NExt}\,\Lambda$ the lattice of its normal extensions. Given $\mathfrak{L} \cong \mathrm{NExt}\,\Lambda$ and some element x in $\mathfrak{L}$, which is the logic of a finite, rooted frame $\mathfrak{F}$, how much can we say about $\mathfrak{F}$?

Notice that this question makes sense even if $\mathfrak{F}$ is not uniquely determined. For example, with Λ an extension of **S4** one always determine the number of elements of $\mathfrak{F}$, independent of whether the theory of $\mathfrak{F}$ is invariant under all automorphisms. We say that the cardinality of $\mathfrak{F}$ is a *lattice constructible function* in NExt **S4**. Or, given x we can effectively determine whether $\mathfrak{F}$ contains a proper cluster. We say therefore that *contains a proper cluster* is a *lattice definable property* in NExt **S4**. It turns out that in order to determine the structure of the group of automorphisms of a lattice we have to study quite carefully which properties are lattice definable or which functions are lattice constructable. We will show for example that any automorphism of NExt **S4** must fix all elements of finite codimension, by establishing enough lattice definable properties and functions so that $\mathfrak{F}$ can be recovered uniquely.

The paper is structured as follows. In §2 we introduce some basic notions and facts about the lattices of normal modal logics and in §3 we establish some results about the groups of automorphisms of these lattices. §4 contains two major results: the first is that a logic containing **S4** of finite codimension is an extension of a pretabular logic iff its lattice of extensions is linear (and finite). The second is that the pretabular logics and all their extensions are pointwise fixed by any automorphism of NExt **S4**. The next section, §5, introduces the notions of lattice definable properties and lattice definable functions and establishes that cardinality, fatness, depth and weight are all lattice definable functions in NExt **S4**. In §6 we show that the lattice NExt **S4.3** has exactly two automorphisms and in §7 that the lattice of logics of finite codimensions extending **S4** has only one automorphism. In §8 we turn to the lattice of extensions of $\mathbf{K.alt}_1$. Its group of automorphisms is proved to be isomorphic to the symmetric goup over the set of natural numbers. Furthermore, for many finite groups G we will construct logics Λ such that the group of automorphisms of $\mathrm{NExt}\,\Lambda$ is isomorphic to G. We end the paper with some open problems.

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2. Lattices and Locales

A structure $\mathfrak{L} = \langle L, \sqcap, \bigsqcup \rangle$, where $\sqcap$ is a binary and $\bigsqcup$ an infinitary operation, is called a **locale** if it is a complete lattice that satisfies the following distributivity law

$$x \sqcap \bigsqcup_{i \in I} y_i = \bigsqcup_{i \in I} x \sqcap y_i$$

A locale is therefore distributive as a lattice. Examples of locales are the open sets of a topological space together with the operations of intersection and infinitary union. A locale is **continuous** if it also satisfies the dual law

$$x \sqcup \bigsqcap_{i\in I} y_i = \bigsqcap_{i\in I} x \sqcup y_i$$

There is a representation theory for locales. The background can be found in [5] and, for modal logic, in [6]. Let $I(\mathfrak{L})$ be the set of all meet–irreducible elements. (An element x is meet-irreducible if from $x = y \sqcap z$ follows $x = y$ or $x = z$.) Now let $x^\dagger := \{y \in I(\mathfrak{L}) : y \not\geq x\}$. Then the following holds:

1. $(x \sqcap y)^\dagger = x^\dagger \cap y^\dagger$
2. $(\bigsqcup_{i\in I} y_i)^\dagger = \bigcup_{i\in I} y_i^\dagger$

Put $\mathfrak{Spec}(\mathfrak{L}) := \langle I(\mathfrak{L}), \{x^\dagger : x \in L\}\rangle$. $\mathfrak{Spec}(\mathfrak{L})$ is a topological space and called the **spectrum** of $\mathfrak{L}$. $\mathfrak{Spec}(\mathfrak{L})$ is a T_0–space. (A space is a T_0–space if for each pair x and y, if x and y are distinct there exists an open set that contains exactly one of them (though we may not be able to choose which one).) The closure of a set $\{x\}$ is exactly the set $I(\mathfrak{L}) - x^\dagger = \{y \in I(\mathfrak{L}) : y \geq x\}$. For $x^\dagger$ is the largest open set not containing x. Given a topological space $\mathfrak{X}$, the open sets form a locale, denoted by $\Omega(\mathfrak{X})$. $\mathfrak{L}$ is called **spatial** if $\mathfrak{L} \cong \Omega(\mathfrak{Spec}(\mathfrak{L}))$. The locales of (normal) extensions of a given modal logic are always spatial (see [6]). The elements of $I(\mathfrak{L})$ are ordered by $\leq$. Moreover, given a T_0–space, we can define a relation $x \leq_t y$ by $y \in \overline{\{x\}}$. This is a partial order, as is easily verified. It turns out that $\leq_t\ =\ \leq$. For $x \leq_t y$ iff $y \in \overline{\{x\}}$ iff $y \geq x$, by the remarks above. Therefore, we will in sequel not distinguish between the order derived from the lattice and the topological order. Now let us look at the connection between the order and the topology. We have seen that the open sets of the topology are lower closed sets. The set of all lower closed sets is a topology, called the **Alexandrov topology**. However, this is not necessarily the only topology that can be defined on a given order. An example will appear below in the last section. It is easy to see that a locale is continuous iff the spectrum carries the Alexandrov topology.

The following is clear. If $\alpha : \mathfrak{L} \to \mathfrak{L}$ is an automorphism then there is a unique automorphism induced on $\mathfrak{Spec}(\mathfrak{L})$, which we will also denote by α. Likewise, an automorphism on a topological space $\mathfrak{X}$ induces a unique isomorphism on $\Omega(\mathfrak{X})$. So, automorphisms of spatial locales can be studied via the automorphisms of their spectrum.

An element x is called **strictly meet–irreducible** if from $x = \bigsqcap_{i\in I} y_i$ follows that $x = y_i$ for some $i \in I$. x is **strictly meet–prime** if from $x \geq \bigsqcap_{i\in I} y_i$ follows $x \geq y_i$ for some $i \in I$. Dually, the notions **strictly join–irreducible** and **strictly join–prime** are defined. In what follows, we call an element **irreducible** (**prime**) if it is strictly meet–irreducible (strictly meet-prime), and **coirreducible** (**coprime**) if it is strictly join–irreducible (strictly join–prime). In a locale, an element is coprime iff it is coirreducible. And a prime element is also irreducible, but the converse does not hold in general. We denote by $Pr(\mathfrak{L})$ the set of primes and by $\mathfrak{Pr}(\mathfrak{L}) := \langle Pr(\mathfrak{L}), \leq\rangle$ the poset of primes. Likewise, $CPr(\mathfrak{L})$ is the set of coprimes and $\mathfrak{CPr}(\mathfrak{L})$ the poset of coprimes. A **splitting pair** is a pair $\langle x, y\rangle$ such that L is the disjoint union of the filter generated by y and the ideal generated by x. In other words, every element is either below x or above y, but not both. The following holds.

Proposition 2.1. *Let $\mathfrak{L}$ be a complete lattice.*

1. *If $\langle x, y\rangle$ is a splitting pair, then x is prime and y is coprime.*
2. *If x is prime there exists a unique coprime y such that $\langle x, y\rangle$ is a splitting pair.*
3. *If y is coprime there exists a unique prime x such that $\langle x, y\rangle$ is a splitting pair.*
4. *If $\langle x, y\rangle$ and $\langle x', y'\rangle$ are splitting pairs then $x \leq x'$ iff $y \leq y'$.*

Let x be prime and y the unique element such that $\langle x, y\rangle$ is a splitting. Then y is called the **splitting companion** of x and denoted by $\mathfrak{L}/x$.

Proposition 2.2. *$x \mapsto \mathfrak{L}/x : \mathfrak{Pr}(\mathfrak{L}) \to \mathfrak{CPr}(\mathfrak{L})$ is an isomorphism of posets.*

Clearly, an automorphism of a locale induces an automorphism of $\mathfrak{Pr}(\mathfrak{L})$. However, automorphisms of $\mathfrak{Pr}(\mathfrak{L})$ may exist without there being a corresponding automorphism of $\mathfrak{L}$. However, if the locale is continuous, automorphisms of $\mathfrak{Pr}(\mathfrak{L})$ are in one–to–one correspondence with automorphisms of $\mathfrak{L}$ itself, for any automorphism of $\mathfrak{L}$ sends lower closed sets onto lower closed sets.

An element x is a **lower cover** of y if $x < y$ and for no z, $x < z < y$. In that case, y is also called an **upper cover** of x. There is an important characterization of cosplittings. It uses the **cocovering number** of x, which is the cardinality of the set of cocovers of x.

Proposition 2.3. *x is cosplitting element of $\mathfrak{L}$ iff (1) x has the cocovering number 1 and (2) for the unique cocover y of x holds that if $z < x$ then $z \leq y$.*

Proof. x is a cosplitting element iff it is coprime iff it is coirreducible. So, we nust show that x is coirreducible iff it satisfies (1) and (2). Let x be coirreducible. Then x cannot have two cocovers, for their join would be x. Therefore, there is a unique cocover. Call it y. If there is an element z such that $z < x$ but $z \not\leq y$ then also $z \sqcup y = x$. So, (1) and (2) hold. Now assume that (1) x has a unique cocover, y, and that (2) for all z with $z < x$ we have $z \leq y$. Then x is coirreducible. For assume $x = \bigsqcup Z$ but $z \neq x$ for all $z \in Z$. Then $z \leq y$ for all $z \in Z$, and so $x > \bigsqcup Z$. Contradiction. So, x is coirreducible. ☐

Now, for a set N of splitting elements we put

$$\mathfrak{L}/N := \bigsqcup \langle \mathfrak{L}/x : x \in N \rangle$$

N is called **independent** or an **antichain** if for all elements $x, y \in N$: if $x \leq y$ then $x = y$.

Proposition 2.4. *Let N be an independent set of splitting elements of cardinality κ. Then $\mathfrak{L}/N$ has cocovering number κ.*

Proof. Let N be independent. Then for each $x \in N$ the element $x \sqcap \mathfrak{L}/N$ is a lower cover of $\mathfrak{L}/N$. These lower covers are different. For let $x, y \in N$ and $x \sqcap \mathfrak{L}/N = y \sqcap \mathfrak{L}/N$. Then $x \sqcap \mathfrak{L}/N \leq y$. But since $x \not\leq y$ and $\mathfrak{L}/N \not\leq y$, we have $x \sqcap \mathfrak{L}/N \not\leq y$, since y is prime. ☐

3. Isomorphisms of Lattices of Logics

Given a modal logic, Λ, the normal extensions of Λ form a locale, denoted by $\mathrm{NExt}\,\Lambda$. (The results of this section do not depend on the language. They carry over to classical logic, intermediate logics, relevance logics and so on.) For a proof of this fact see [6]. In this section we want to consider briefly the correspondence between automorphisms of the lattice of extensions of some logic and bijections of the language. Before we do so, we need to emphasize that if this bijection is required to be a homomorphism (ie a substitution) this correspondence turns out to be trivial. Suppose that we are given a language L and a bijective homomorphism $\pi : L \to L$. Then π is a substitution, and its inverse is also a substitution. It is easy to see that π is generated by a permutation of the set of variables. In that case, the induced action on $\mathrm{NExt}\,\Lambda$, which we take to be $\Theta \mapsto \pi[\Theta]$, is the identity. Hence, we shall not assume that π is a homomorphism.

Every logic is an intersection of meet–irreducibles. Therefore, the previous representation theorems can be sharpened somewhat by taking instead of the set of meet–irreducibles the set of strictly meet–irreducibles. Let $Ir(\mathfrak{L})$ be the set strictly meet–irreducible elements, and let $\mathfrak{ISpc}(\mathfrak{L})$ be the topological space induced by $\mathfrak{Spec}(\mathfrak{L})$ on $Ir(\mathfrak{L})$. Let

$$x^{\ddagger} := Ir(x) - x^{\dagger}$$

Theorem 3.1. $\Lambda = \bigsqcap \Lambda^{\ddagger}$. *Moreover, $\mathrm{NExt}\,\Lambda \cong \Omega(\mathfrak{ISpc}(\mathfrak{L}))$. Hence, $\mathrm{NExt}\,\Lambda$ is spatial.*

The following theorem underlines the thesis that a logic is — in a sense still to be defined — determined by its lattice of extensions.

Theorem 3.2. *Let Λ and Λ' be two modal logics in the modal language L. $NExt\,\Lambda \cong NExt\,\Lambda'$ iff there exists a bijection $\pi : L \to L$ such that for any logic Θ extending Λ, $\pi[\Theta]$ is a logic extending Λ', and for every logic Θ' extending Λ', $\pi^{-1}[\Theta']$ is a logic extending Λ.*

Proof. It is clear that if π has the desired properties then NExt Λ and NExt Λ' are isomorphic. Now assume that the two lattices are isomorphic. Let β : NExt $\Lambda \to$ NExt Λ' be an isomorphism. Call a set S of formulae Λ–**minimal** if it is of the form

$$S(\Theta) := \Theta - \bigcup_{\Delta < \Theta} \Delta$$

for some Θ. Here, Δ ranges over logics containing Λ. The following is observed about minimal sets. (i) If $\Theta \neq \Theta'$ then $S(\Theta) \cap S(\Theta') = \varnothing$. For suppose that $\varphi \in S(\Theta) \cap S(\Theta')$. Then $\varphi \in \Theta \cap \Theta'$, but φ is not in any logic properly contained in Θ or Θ'. Hence $\Theta \cap \Theta'$ cannot be properly contained in Θ or Θ'. So, $\Theta \cap \Theta' = \Theta = \Theta'$. (ii) Each formula is in a minimal set. For let φ be a formula. Let Θ be the intersection of all logics in NExt Λ which contain φ. This is a logic, and its minimal set contains φ. (iii) $S(\Theta) = \varnothing$ iff Θ is the limit of an infinite ascending chain. Otherwise $S(\Theta)$ is countably infinite. For a proof, suppose that Θ is the limit of an infinite ascending chain, say $\Theta = \bigsqcup_{i \in \omega} \Delta_i$. Then $\Theta = \bigcup_{i \in \omega} \Delta_i$, by compactness. Hence, $S(\Theta) = \varnothing$. Now suppose that Θ is not the limit of an infinite ascending chain. Then Θ is finitely axiomatizable relative to Λ. Hence there is a φ such that $\varphi \in \Theta$, but $\varphi \notin \Delta$ for any $\Delta < \Theta$. Now, fix such a φ. The formulae

$$\top \vee \top \vee \ldots \vee \top \to \varphi$$

are then also in $S(\Theta)$. Hence, $S(\Theta)$ is infinite. Since the language is countable, $S(\Theta)$ is countably infinite.

Now define similarly the sets $S'(\Theta)$ for $\Theta \in$ NExt Λ' by

$$S'(\Theta) := \Theta - \bigcup_{\Delta < \Theta} \Delta$$

where now Δ ranges over all extensions of Λ'. $S'(\Theta)$ is empty iff Θ is the limit of an ascending chain. Otherwise $S'(\Theta)$ is countably infinite. For each $\Theta \in$ NExt Λ there exists a bijection $\pi_\Theta : S(\Theta) \to S'(\beta(\Theta))$. Hence, let $\pi := \bigcup \pi_\Theta$ be the union of these bijections. This is well–defined, since the minimal sets are pairwise disjoint. It is a function from L to L' since every formula is in a minimal set. It is injective since the $S'(\beta(\Theta))$ are pairwise disjoint and every π_Θ is injective. Finally, π is surjective. For let φ be given; then $\varphi \in S'(\Delta)$ for some Δ. Since β is an isomorphism, there is a Θ such that $\beta(\Theta) = \Delta$. Since π_Θ is surjective, there is a ψ such that $\pi_\Theta(\psi) = \varphi$. Consequently, $\pi(\psi) = \varphi$. □

Even if $\Lambda \neq \Lambda'$ or $L \neq L'$, π is in general not a homomorphism (that is, a substitution). For suppose it necessarily is. Then π^{-1} is a substitution, and so is $\pi^{-1} \circ \pi$. But we have seen earlier that this map induces the identity. So, there are bijections which are not homomorphisms. Another argument is the following. Suppose that π is a bijection and a homomorphism of the languages. Then π is induced by a bijection between the variables. It follows that Θ has interpolation iff $\pi[\Theta]$ has interpolation. For assume that $\pi[\Theta]$ has interpolation and let $\varphi \to \psi \in \Theta$. Then $\pi(\varphi \to \psi) \in \pi[\Theta]$. Since $\pi(\varphi \to \psi) = \pi(\varphi) \to \pi(\psi)$ we have a χ' such that $var(\chi') \subseteq var(\pi(\varphi)) \cap var(\pi(\psi))$ and $\pi(\varphi) \to \chi', \chi' \to \pi(\psi) \in \pi[\Theta]$. Now put $\chi := \pi^{-1}(\chi')$. Then $var(\chi) \subseteq var(\varphi) \cap var(\psi)$, and $\pi(\varphi \to \chi) = \pi(\varphi) \to \chi' \in \pi[\Theta]$ as well as $\pi(\chi \to \psi) = \chi' \to \pi(\psi) \in \pi[\Theta]$. Hence, $\varphi \to \chi \in \Theta$ and $\chi \to \psi \in \Theta$. So, Θ has interpolation. Exchanging the roles of Θ and $\pi[\Theta]$ we find that if Θ has interpolation then $\pi[\Theta]$ has interpolation as well. Similarly, it is shown that Θ is Halldén–complete iff $\pi[\Theta]$ is Halldén–complete.

The logic $\Theta_1 := \mathbf{K} \oplus \Box\Box\bot$ has interpolation, since it is the extension of $\mathbf{K}$ by a constant formula. (See [9] for a proof.) It is not Halldén–complete. For the formula $\Box\bot \vee \neg\Box\bot$ is a theorem of Θ_1, but neither $\Box\bot$ nor $\neg\Box\bot$ are theorems of Θ_1. The logic of the frame $\langle\{0, 1\}, \leq\rangle$, which we denote by Θ_2, does not have interpolation, but is Halldén–complete. (It is namely identical with the quasi–normal logic of the pointed frame $\langle\{0,1\},\leq,0\rangle$. All such logics are Halldén–complete.) Now $\mathrm{NExt}\,\Theta_1 \cong \mathrm{NExt}\,\Theta_2 \cong \mathbf{3}$. This proves that the map π cannot be a homomorphism in general.

We are interested mainly in the structure of the group of automorphisms of the locales of some modal logics. If $\mathfrak{X}$ is some structure (for example, a locale or a topological space) we write $\mathfrak{Aut}\,\mathfrak{X}$ for the group of automorphisms of $\mathfrak{X}$. $\mathfrak{X}$ is **rigid** if $\mathfrak{Aut}\,\mathfrak{X}$ is the one–element group. Not much group theory is needed to understand the results of this paper. The group of bijections from M to M is denoted by $Sym(M)$. As usual, we choose M to be a cardinal number. The cyclic group of order n is denoted by $\mathbb{Z}_n$. We are interested in automorphisms of structures, notably lattices of logics. If G operates on a structure $\mathfrak{S}$ over a set S, then the set $\{\alpha(x) : \alpha \in G\}$ is called the G–**orbit** of x. An automorphism α of some structure **fixes** an element x if $\alpha(x) = x$. We write $Fix(\alpha)$ for the set of elements fixed by α. α **fixes** a set S if $\alpha[S] = S$. α fixes S **pointwise** if $\alpha(x) = x$ for all $x \in S$ iff $S \subseteq Fix(\alpha)$. It is clear that for example the set of prime elements and the set of coprime elements of a locale are fixed under any automorphism — though not necessarily pointwise. Moreover, if $\langle x, y\rangle$ is a splitting pair then so is $\langle \alpha(x), \alpha(y)\rangle$. By the uniqueness of the splitting companion we deduce the following lemma.

Lemma 3.3. *Suppose that $\alpha \in \mathfrak{Aut}(\mathfrak{L})$ and that x splits $\mathfrak{L}$. $x \in Fix(\alpha)$ iff $\mathfrak{L}/x \in Fix(\alpha)$. Moreover, α fixes the following sets: $\uparrow x$, $\downarrow x$, $\uparrow \mathfrak{L}/x$ and $\downarrow \mathfrak{L}/x$.*

It follows that if α fixes x, its restriction to $\uparrow \mathfrak{L}/x$ is an automorphism of $\mathrm{NExt}\,\mathfrak{L}/x$. An immediate consequence of the previous theorem is the following.

Lemma 3.4. *Suppose that every automorphism of $\mathfrak{L}$ fixes x and x is prime. Then every automorphism fixes $\mathfrak{L}/x$.*

The lattice of extensions of a logic has cardinality $\leq 2^{\aleph_0}$. Many standard logics (**K**, **S4**, **Grz**) have continuously many extensions. At first blush the size of the group of automorphisms can therefore be larger than $2^{\aleph_0}$. This however is not so. This is a corollary of the next theorem.

Proposition 3.5. *An automorphism of $NExt\,\Lambda$ fixes the set of logics which are finitely axiomatizable over Λ.*

There are two proofs of this theorem. A logic is finitely axiomatizable iff it is compact. (x is **compact** if $x \leq \bigsqcup_{i\in I} y_i$, then there exists a finite set $J \subseteq I$ such that $x \leq \bigsqcup_{i\in J} y_i$.) Since the set of compact elements is fixed, so is the set of finitely axiomatizable logics. Clearly, the action of the group on the lattice is completely determined by its action on the compact elements. Now there are only countably many finitely axiomatizable logics. Hence we have the following theorem.

Theorem 3.6. *Let Λ be a logic. Then $\sharp\,\mathfrak{Aut} NExt\,\Lambda \leq 2^{\aleph_0}$.*

We will see that there are logics for which this limit is obtained. So no better bound exists. A second proof consists in the observation that by Theorem 3.2 an automorphism of $\mathrm{NExt}\,\Lambda$ is a factor group of $Sym(L)$.

Theorem 3.7. *Let $St(\Lambda)$ be the group of all permutations of L that induce the identity on $NExt\,\Lambda$. $St(\Lambda)$ is a normal subgroup of $Sym(L)$, and $\mathfrak{Aut}\ NExt\,\Lambda \cong Sym(L)/St(\Lambda)$.*

We will draw some conclusions from these facts. Blok has shown that each of the two logics of codimension 1 in the lattice of normal modal logics has $2^{\aleph_0}$ cocovers. These logics of codimension 2 all have the same lattice of extensions, namely **3**. However, not every permutation of these logics is induced by an automorphism. The reason is simple: an automorphism must send a finitely axiomatizable logic to a finitely axiomatizable logic. This will be rephrased as follows. Let

$$C_\Theta(\Lambda) := \{\Lambda' : \textit{for some } \alpha \in \mathfrak{Aut}(\mathrm{NExt}\,\Theta) : \alpha(\Lambda') = \Lambda\}$$

Call $C_\Theta(\Lambda)$ the **l–spectrum** of Λ with respect to Θ and $\sharp C_\Theta(\Lambda)$ the **l–indeterminacy** of Λ with respect to Θ. (The l–spectrum is nothing but the orbit of Λ under the group of automorphisms of $\mathrm{NExt}\,\Theta$.)

Lemma 3.8. *Let Λ be finitely axiomatizable over Θ. Then the l–indeterminacy of Λ with respect to Θ is countable.*

4. Getting Started

A **Kripke–frame** is a pair $\mathfrak{F} = \langle F, \lhd \rangle$, where F is a set and $\lhd \subseteq F^2$. We assume here always that F is nonempty. In sequel, a **frame** is always understood to be a **Kripke–frame**. $\mathfrak{G} = \langle G, \lhd_G \rangle$ is a **generated subframe** of $\mathfrak{F}$ if $G \subseteq F$ and $\lhd_G = \lhd \cap G^2$. $\mathfrak{F}$ is **rooted** if there is a point $x \in F$ such that $\mathfrak{F}$ is the smallest generated subframe of $\mathfrak{F}$ containing x. We assume familiarity with the usual concepts such as p–morphism. Given two frames $\mathfrak{F}$ and $\mathfrak{G}$ we write $\pi : \mathfrak{F} \to \mathfrak{G}$ iff π is a p–morphism from $\mathfrak{F}$ into $\mathfrak{G}$. If π is onto we write $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$. Put $\mathsf{Th}\,\mathfrak{F} := \{\varphi : \mathfrak{F} \models \varphi\}$. If $\Theta = \mathsf{Th}\,\mathfrak{F}$ for some finite rooted frame $\mathfrak{F}$, we say that Θ is **tabular** and that $\mathfrak{F}$ a **generating frame** of Θ. The next theorem asserts that this frame is unique up to isomorphism.

Proposition 4.1. *Assume that $\mathfrak{F}$ and $\mathfrak{G}$ are finite rooted frames. Then $\mathsf{Th}\,\mathfrak{F} = \mathsf{Th}\,\mathfrak{G}$ iff $\mathfrak{F} \cong \mathfrak{G}$.*

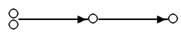
Proof. By Jónsson's Theorem. We prove it for algebras, which is the same in the case of finite structures. We have $\mathsf{Th}\,\mathfrak{A} = \mathsf{Th}\,\mathfrak{B}$ iff $\mathfrak{A} \in \mathsf{HSP}\,\mathfrak{B}$ and $\mathfrak{B} \in \mathsf{HSP}\,\mathfrak{A}$ iff $\mathfrak{A} \in \mathsf{HSP}_u\,\mathfrak{B}$ and $\mathfrak{B} \in \mathsf{HSP}_u\,\mathfrak{A}$ iff $\mathfrak{A} \in \mathsf{HS}\,\mathfrak{B}$ and $\mathfrak{B} \in \mathsf{HS}\,\mathfrak{A}$ iff $\mathfrak{A} \cong \mathfrak{B}$. ☐

This means in particular, that we may study the action of α on the set of rooted finite frames modulo isomorphism. Namely, if $\Lambda = \mathsf{Th}\,\mathfrak{F}$, $\Theta = \mathsf{Th}\,\mathfrak{G}$ and $\alpha(\Lambda) = \Theta$, then we also write $\alpha(\mathfrak{F}) = \mathfrak{G}$. Notice that this is uniquely defined only if there is only one frame from each isomorpism class.

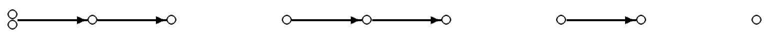
In what follows, we will write $\Lambda/\mathfrak{F}$ in place of $\mathrm{NExt}\,\Lambda/\mathsf{Th}\,\mathfrak{F}$. Now let $\mathfrak{L} = \langle L, \leq \rangle$ be a lattice, and $x \in L$. It follows, by a theorem of Blok ([2]), that an automorphism must fix the set of logics of rooted, finite, cycle–free frames. Furthermore, by a theorem of Makinson ([7]), $\mathrm{NExt}\,\mathbf{K}$ has only two coatoms. Only one of them is a prime element.

Proposition 4.2. *An automorphism of NExt* $\boldsymbol{K}$ *fixes the set of coatoms pointwise.*

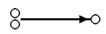
Given a partial order $\leq$ we write $x \prec y$ if x is a lower cover of y. x has **codimension** n if the longest properly ascending chain starting at x has $n + 1$ members. However, we generally look at a different codimension of x, namely in the poset of coirreducibles. We call this its *order codimension*. Λ has **order codimension** n if the longest ascending chain of coirreducible logics starting at Λ has length n. (We note that maximal chains need not be of equal length. Therefore we take the order codimension to be the length not of a maximal chain but of a chain of maximal length, ie a longest chain.) It might be deemed that the codimension is 1 + the order codimension. However, the situation is more complicated. For look at the following frame.



Its order codimension is 3, for the following chain can be constructed.



However, the logic has other extensions as well, for example based on the frame



Therefore the codimension of this logic is 5, which is greater than 1 + 3. x has **order covering number** n if x has exactly n irreducible covers, and **order cocovering number** n if it has exactly n irreducible cocovers. The reason for taking these numbers rather than the ordinary covering and cocovering numbers lies in the fact that irreducible logics of finite codimension correspond to finite rooted frames (at least in $\mathrm{NExt}\,\mathbf{K4}$). Therefore, we do not measure how many covers or cocovers an element has in the lattice but rather how many there are in the partial order of irreducible elements. Usually, these numbers are studied with x in a sublattice of the form $\uparrow y$. The order covering number of x does not depend on the choice of y; however, the order cocovering number of x does. It is clear that if $\alpha : L \to L$ is an automorphism of a lattice, then $\alpha(x)$ has the same codimension, the same covering number and the same cocovering number as x.

Below we will focus on **S4**–logics. Hence, let us review some basic facts and terminology for them. The following is folklore.

Theorem 4.3. *Let $\Theta \in NExt\,\mathbf{S4}$. Then Θ is tabular iff it is of finite codimension.*

Let $\mathfrak{F} = \langle F, \lhd \rangle$ be a reflexive transitive frame. A subset $C \subseteq F$ is called a **cluster** of $\mathfrak{F}$ if it is of the form $\{y : x \lhd y \lhd x\}$. $\sharp C$ is called the **fatness** of C and $ft(\mathfrak{F}) := max\{\sharp C : C \text{ a cluster of } \mathfrak{F}\}$ the **fatness** of $\mathfrak{F}$. We shall in general not distinguish between the cluster C and the frame $\langle C, C^2 \rangle$ defined by it. The latter type of frames is in fact also called a **cluster**. $\mathfrak{F}$ is **slender** if it is of fatness 1. **Grz** is the logic of finite slender frames. C is **proper** if it has fatness > 1, otherwise it is improper. In a finite transitive frame $\mathfrak{F}$, the depth of a point, $dp(x)$, is defined by

$$dp(x) := \{dp(y) : x \lhd y \not\lhd x\}$$

This means the following. A frame is of depth 0 if it is in a final cluster. x is of depth $n+1$ if it has successors of depth n and every successor y of x is either in the same cluster or of depth $\leq n$. The depth of the frame $\mathfrak{F}$ is defined as $dp(\mathfrak{F}) := \{dp(x) : x \in f\}$. So, the frame above has a two point cluster of depth 2, and two improper clusters of depth 1 and 0. The depth of the frame is 3, by definition.

For an extension Θ of **S4** we can show that Θ is irreducible and has order codimension n iff Θ is the logic of a rooted n–point **S4**–frame. This follows from the following fact.

Lemma 4.4. *Let $\mathfrak{F}$ be a rooted $n+1$–point* **S4***–frame. Then there exists a rooted n point* **S4***–frame $\mathfrak{G}$ and a p–morphim from $\mathfrak{F}$ onto $\mathfrak{G}$.*

Proof. Look at the set T of final clusters. Case (A). There is a proper cluster C in T. Then two points in C can be identified, reducing C by one point. Case (B). T has two elements, C and D, both improper. Then collapsing D and C is a p–morphism reducing the number of points by 1. Case (C). T has one member only, C, and C is improper. Then if $\mathfrak{F}$ has at least two points, there exists a cluster D immediately preceding C. If D is proper, we proceed as in (A). So, assume that D is improper. Then collapsing D and C is a p–morphism, reducing the number of points by 1. ☐

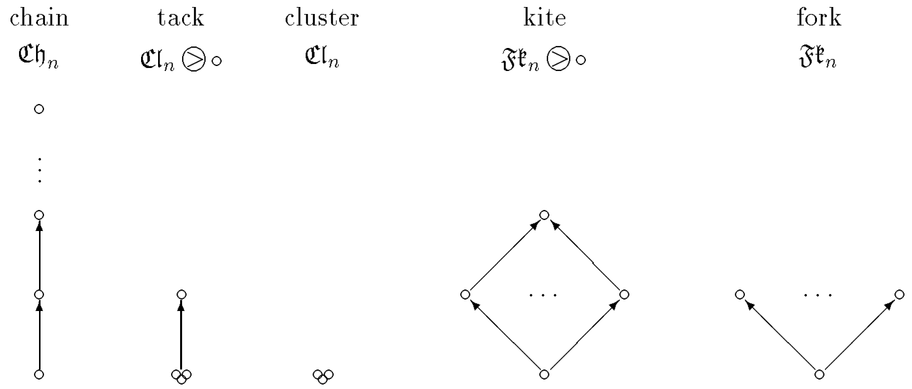
Lemma 4.5. *Let $\mathfrak{F}$ be a rooted $\mathbf{S4}$–frame of cardinality n. Then $\mathsf{Th}\,\mathfrak{F}$ has order codimension n in $NExt\,\mathbf{S4}$. Let Λ, $\Lambda' \in NExt\,\mathbf{S4}$. Let $\alpha : NExt\,\Lambda \to NExt\,\Lambda'$ be an isomorphism. Suppose that $\Theta \in NExt\,\Lambda$ is the logic of an n point rooted frame. Then $\alpha(\Theta)$ too is the logic of an n point rooted frame.*

We will introduce some further notation. Given two frames, $\mathfrak{F} = \langle F, \lhd_F \rangle$ and $\mathfrak{G} = \langle G, \lhd_G \rangle$, we write $\mathfrak{F} \oslash \mathfrak{G}$ for the frame obtained by placing $\mathfrak{F}$ before $\mathfrak{G}$. It is defined formally as follows.

$$\begin{array}{rcl} \mathfrak{F} \oslash \mathfrak{G} & := & \langle f \times \{0\} \cup g \times \{1\}, \lhd^+ \rangle \\ \lhd^+ & := & \left\{ \begin{array}{ll} & \{\langle \langle x, 0 \rangle, \langle y, 0 \rangle \rangle : x, y \in F, x \lhd_F y\} \\ \cup & \{\langle \langle x, 0 \rangle, \langle y, 1 \rangle \rangle : x \in F, y \in G\} \\ \cup & \{\langle \langle x, 1 \rangle, \langle y, 1 \rangle \rangle : x, y \in G, x \lhd_G y\} \end{array} \right. \end{array}$$

We will use Lemma 4.5 to show that automorphisms must fix certain elements in the lattice. We say that a rooted frame $\mathfrak{F}$ has **covering number** n if there are exactly n rooted frames $\mathfrak{G}$ such that $\mathsf{Th}\,\mathfrak{G}$ covers $\mathsf{Th}\,\mathfrak{F}$. Analogously the **cocovering number** of $\mathfrak{F}$ is defined. A logic is **pretabular** if it is not tabular, but all its proper extensions are. Recall that **S4** has five pretabular systems (see [8]). The first is **S5**. It is the logic of the clusters; the n–point cluster is denoted here by $\mathfrak{Cl}_n$. The second is the logic of the **tacks**; the $n+1$–point tack is $\mathfrak{Cl}_n \oslash \circ$. The third is the logic **Grz.3**. It is the logic of all chains. The n–point chain is denoted by $\mathfrak{Ch}_n$. The fourth is the logic of the **forks**. $\mathfrak{Fk}_n$, where n is the number of points of depth 0, is the $n+1$–point fork. And the fifth is the logic of the **kites**. The $n+2$–point kite is $\mathfrak{Fk}_n \oslash \circ$. (See Figure 1.) There are a few isomorphisms:

FIGURE 1. The five pretabular extensions of **S4**

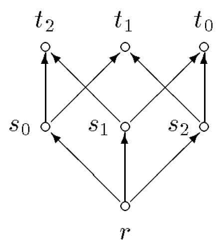


$\mathfrak{Cl}_1 \cong \mathfrak{Ch}_1$, $\mathfrak{Ch}_2 \cong \mathfrak{Cl}_1 \ogreaterthan \circ \cong \mathfrak{Fk}_1$, $\mathfrak{Fk}_1 \ogreaterthan \circ \cong \mathfrak{Ch}_3$. We call a frame a **handle** if it is one of the above, ie a cluster, a tack, a chain, a fork or a kite.

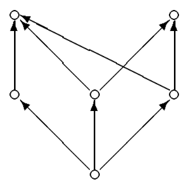
Lemma 4.6. *Each handle has covering number 1.*

The converse does not hold. There are frames with covering number 1 which are not handles. An example are the frames of [4]. These are defined as follows.

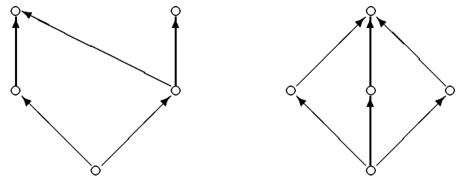
$$\begin{array}{rcl} U_n & := & \{r\} \cup \{s_i : i < n\} \cup \{t_i : i < n\}, \\ \lhd_n & := & \quad \{\langle x, x\rangle : x \in U_n\} \cup \{\langle r, x\rangle : x \in U_n\} \\ & & \cup \{\langle s_i, t_j\rangle : i \neq j\} \\ \mathfrak{U}_u & := & \langle U_n, \lhd_n\rangle \end{array}$$



These frames have covering number 1. For a proof, suppose that $\mathfrak{G}$ is a cover of $\mathfrak{U}_n$. Case A. There is a p–morphism $\pi : \mathfrak{U}_n \to \mathfrak{G}$. Then π collapses exactly two points, say x and y. It is easy to see that x and y must be of same depth, and that this depth is 0. Moreover, for any pair x' and y' of depth 0 there is an automorphism of $\mathfrak{U}_n$ mapping x to x' and y to y'. So, $\mathfrak{G}$ is unique up to isomorphism. Case B. $\mathfrak{G}$ is a generated subframe of $\mathfrak{U}_n$. Then it is the fork $\mathfrak{Fk}_{n-1}$. But $\mathfrak{Fk}_{n-1}$ is also a generated subframe of the frame obtained in Case A. So, $\mathfrak{U}_n$ has only one cover. For example, the frame $\mathfrak{U}_3$ has the following unique cover:



However, this cover has the following frames as p–morphic images



This is no coincidence. For the following can be shown. (Notice that the proof makes use of the classification of cocovers, established below.)

Lemma 4.7. *Let $\mathfrak{F}$ be a finite rooted* **S4***–frame such that NExt*$\mathsf{Th}\,\mathfrak{F}$ *is linear. Then $\mathfrak{F}$ is a handle.*

Proof. We prove the claim by induction on the size of $\mathfrak{F}$. If it is 1 or 2, we are done, since $\mathfrak{F}$ is a handle. Now suppose that $\mathfrak{F}$ has at least 3 elements, and that NExt $\mathsf{Th}\,\mathfrak{F}$ is linear. Then for every $\mathfrak{G}$ such that $\sharp\mathfrak{G} < \sharp\mathfrak{F}$, NExt $\mathsf{Th}\,\mathfrak{G}$ is also linear. By induction hypothesis, $\mathfrak{G}$ is a handle. It follows that $\mathfrak{F}$ has covering number 1 and the unique cover, $\mathfrak{G}$, is a handle. We can on this fact alone exclude the case that $\mathfrak{F}$ has two proper clusters. For $\mathfrak{G}$ has at most one proper cluster, and therefore this case can only arise if $\mathfrak{G}$ is a tack. Then $\mathfrak{F} \cong \mathfrak{Cl}_n \ogreaterthan \mathfrak{Cl}_2$. Then $\mathfrak{F}$ has two covers, $\mathfrak{Cl}_n \ogreaterthan \circ$ and $\mathfrak{Cl}_{n-1} \ogreaterthan \mathfrak{Cl}_2$. Contradiction. So, at most one cluster is proper.

Now assume that $\mathfrak{G}$ is a cluster, say $\mathfrak{G} \cong \mathfrak{Cl}_n$, $n > 1$. Then $\mathfrak{F} \cong \mathfrak{Cl}_{n+1}$ or $\mathfrak{F} \cong \circ \ogreaterthan \mathfrak{Cl}_n$. The latter case cannot arise, however, since in that case $\mathfrak{F}$ has two covers, $\circ \ogreaterthan \mathfrak{Cl}_{n-1}$ and $\mathfrak{Cl}_n$, contrary to our assumption.

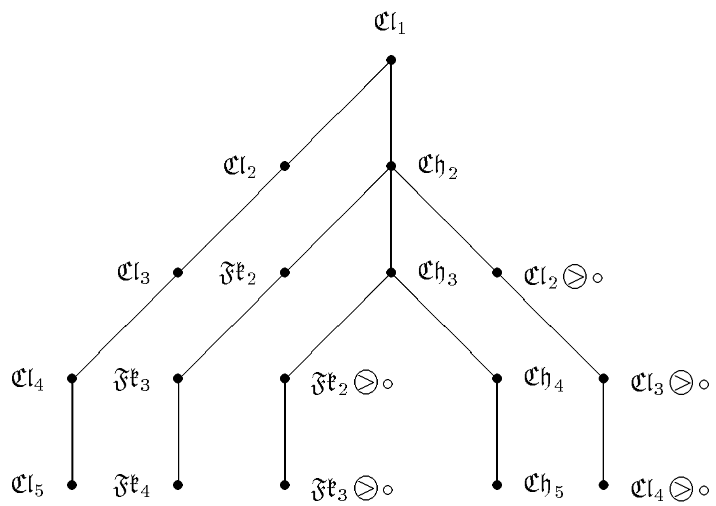
Next assume that $\mathfrak{G}$ is a tack, say $\mathfrak{G} \cong \mathfrak{Cl}_n \ogreaterthan \circ$, $n > 0$. Then $\mathfrak{F}$ is isomorphic to either of the following frames: $\circ \ogreaterthan \mathfrak{Cl}_n \ogreaterthan \circ$, $\mathfrak{Cl}_{n+1} \ogreaterthan \circ$, $\mathfrak{Cl}_n \ogreaterthan \circ \ogreaterthan \circ$, $\mathfrak{Cl}_n \ogreaterthan \mathfrak{Cl}_2$ or $\mathfrak{Cl}_n \circ (\circ \oplus \circ)$. (Here, $\circ \oplus \circ$ is the disjoint sum of two improper clusters.) It is readily checked that $\mathfrak{F}$ has two covers except when it is isomorphic to a tack.

Now we assume that $\mathfrak{G}$ is a chain, a kite or a fork. In particular it has no proper cluster. We will show first that also $\mathfrak{F}$ has no proper cluster. Suppose for the sake of contradiction that it does. Then the proper cluster is of size 2. Let it be $\{x, y\}$. $\mathfrak{G}$ is obtained from $\mathfrak{F}$ by dropping from this cluster one point, say y. Now, there is a p–morphism $\pi : \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ onto some (unique) cover of $\mathfrak{G}$, $\mathfrak{H}$. Expand the cluster $\pi(x)$ by adding some point, z. This defines the frame $\mathfrak{H}^+$. Define $\pi^+ : \mathfrak{F} \twoheadrightarrow \mathfrak{H}^+$ by putting $\pi^+(y) := z$ and $\pi^+(x) := \pi(x)$ else. This is easily seen to be a p–morphism. Now, $\mathfrak{G}$ is not isomorphic to $\mathfrak{H}^+$, but both have the same cardinality, namely $\sharp\mathfrak{F} - 1$. So, $\mathfrak{F}$ has two covers, a contradiction.

Therefore $\mathfrak{F}$ has no proper clusters. We consider $\mathfrak{F}$ as the result of adding a point x to $\mathfrak{G}$. $\mathfrak{G}$ is either a chain, a kite or a fork. Suppose that it is a chain and of depth at least 4. (The case that $\mathfrak{G}$ has depth 2 is covered by the case where $\mathfrak{G}$ is a tack, and in case the depth is 3, $\mathfrak{G}$ is also a kite. This case will be dealt with below.) If $\mathfrak{F}$ is not also a chain then x is not seen by all members of $\mathfrak{G}$. Let I be the set of members not seeing x. If I has more than two members, it has two members y_1 and y_2 such that y_2 immediately succeeds y_1. Collapsing y_1 and y_2 is a p–morphism producing a cover of $\mathfrak{F}$ that is not a handle. Contradiction. So I has one member. So the complement of I contains two points y_1 and y_2 such that y_2 immediately succeeds y_1. Collapsing y_1 and y_2 is a p–morphism onto some frame that is not a handle. Contradiction. So $\mathfrak{F}$ is also a chain.

Now suppose that $\mathfrak{G}$ is a kite. Then it is easy to see that x is not at the root of $\mathfrak{F}$ and therefore not of depth > 1. If it is of depth 1 then $\mathfrak{F}$ is already a kite. So assume for sake of contradiction that x has depth 0. Let y be the other point of depth 0. Case 1. There is a point z seeing only x. Then collapsing x and z is a p–morphism onto some frame that is not a handle. Contradiction. Case 2. There is a point seeing only y. Similarly. Case 3. All points see both x and y. There are at least two points of depth 1. Let u and v be such points. Collapsing u and v is a p–morphism onto some frame that is not a handle. Contradiction.

FIGURE 2. The irreducibles of NExt **Hd**



Finally, assume that $\mathfrak{G}$ is a fork. As in the previous case we can see that x is not of depth 2 or 1. So it is of depth 0, and $\mathfrak{F}$ is a fork. $\square$

(I am indebted to the referee for pointing out this idea of proof.) Since this is a rather remarkable fact, we restate it once again. (We use the notation $\mathfrak{L}^\perp$ for the lattice dual to $\mathfrak{L}$. For example, $\omega^\perp = \langle \omega, \geq \rangle$.)

Theorem 4.8. *(a) An **S4**–logic properly contains a pretabular logic iff its lattice of extensions is finite and linear. (b) An **S4**–logic is pretabular iff its lattice of extensions is isomorphic to $\omega^\perp$, the order dual to ω.*

From this fact we derive first of all that an automorphism of NExt **S4** leaves the set of handles invariant. However, a closer look at the matter reveals that Theorem 4.8 is not needed. This follows namely directly from the fact that a logic is tabular iff it is of finite codimension. A logic is therefore pretabular iff it has codimension ω. (In the infinite case, the codimension is not always defined, but in this case it is.) We deduce that any automorphism must send pretabular logics to pretabular logics. Hence it fixes the sets of handles which contain handles of same cardinality. There are at most five of them.

Let **Hd** be the logic of handles. **Hd** is uniquely defined as the smallest logic whose tabular extensions all have linear extension lattices. $\mathfrak{Aut}$ NExt **Hd** is a subgroup of $Sym(5)$. This group is rather large. However, in fact NExt **Hd** has far less automorphisms. This follows from the fact that the partial order of handles is not the disjoint sum of 5 linear orders of type $\omega^\perp$. Its structure is more complex. The proof of the next result can in fact be deduced immediately from looking at the upper part of the poset, as depicted in Figure 2.

Proposition 4.9. $\mathfrak{Aut}$*(NExt **Hd**) $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, every automorphism of NExt **Hd** fixes the clusters pointwise.*

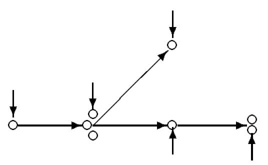
Proof. NExt **Hd** has exactly two elements of codimension 2, the two element chain and the two element cluster. Since the tacks, the chains, the forks and the kites are below the two element chain, every automorphism of NExt **Hd** fixes the clusters. There are three elements of codimension

3 below the chain $\mathfrak{Ch}_2$. These are the chain $\mathfrak{Ch}_3$, the fork $\mathfrak{Fk}_2$ and $\mathfrak{Cl}_2 \otimes \circ$. The chains and the kites are below $\mathfrak{Ch}_3$. Automorphisms can send kites to chains, but not to tacks or forks. Therefore, the set of forks and tacks, and the set of kites and chains are each fixed, though not necessarily pointwise. Hence the group of automorphisms must be a subgroup of the direct product of $\mathbb{Z}_2$ with itself. We show that it is exactly that group. Let α be the map that sends the tack $\mathfrak{Cl}_k \otimes \circ$ to the fork $\mathfrak{Fk}_k$, and the fork $\mathfrak{Fk}_k$ to the tack $\mathfrak{Cl}_k \otimes \circ$, and is the identity elsewhere. This is an automorphism of the poset of handles, and therefore of NExt **Hd**. α is an involution. Let β be the map that sends the chain $\mathfrak{Ch}_{k+2}$ to the kite $\mathfrak{Fk}_k \otimes \circ$, and the kite $\mathfrak{Fk}_k \otimes \circ$ to the chain $\mathfrak{Ch}_{k+2}$ ($k > 1$), and fixes all other frames. Then β is an automorphism of order 2. It commutes with α. ☐

We will also show that every automorphism of NExt **S4** fixes **Hd** and therefore fixes NExt **Hd** pointwise. To this end we look at the order cocovering numbers. Actually, we only need to establish that the forks cannot be mapped onto the tacks and that the chains cannot be mapped onto the kites. We will prove a little bit more here by computing all cocovering numbers of the handles. Below in the picture it is shown where in the tack $\mathfrak{Cl}_2 \otimes \circ$ points can be inserted. (So, one sees in total 8 points, three from the original frame and 5 for the possible insertion points. This shows that the cocovering number is 5.)

Lemma 4.10. *The clusters have order cocovering number 2.*

Proof. Suppose we add somewhere a point to get a rooted frame $\mathfrak{G}$ such that $\mathfrak{F}$ is a p–morphic image or a generated subframe of $\mathfrak{G}$. We may then either add a point at depth 1, or increase the cluster by 1. (We may not place a point following the cluster, for we would get the frame $\mathfrak{Cl}_k \otimes \circ$, which cannot be mapped onto $\mathfrak{Cl}_k$, except when $k = 1$. In that case we get the frame $\circ \otimes \circ$, which also results from $\circ$ by placing the point *before* the cluster.) ☐



Lemma 4.11. *The tacks have order cocovering number 5.*

Proof. We may add a point to the cluster, before it, after it, we may increase the final cluster, and we may add a cluster at depth 1. (See also the picture. The points with an arrow are possible places of insertion. The frame under consideration is $\mathfrak{Cl}_2 \otimes \circ$. Therefore we have three points without an arrow pointing to them, and five more points.) ☐

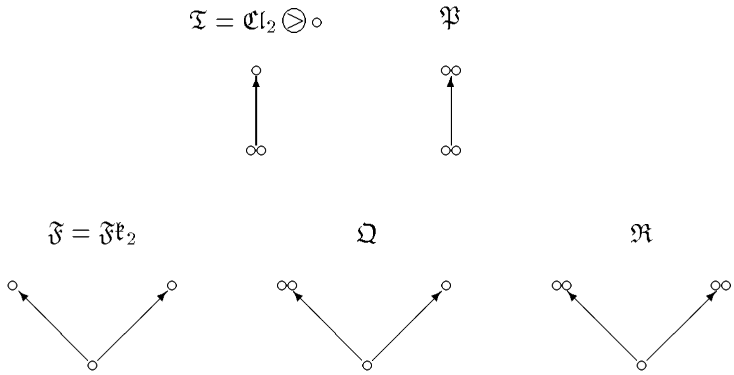
Lemma 4.12. *The chain $\mathfrak{Ch}_n$ has order cocovering number $n + \binom{n+1}{2}$.*

Proof. (a) We may increase each cluster by 1. (n possibilities.) (b) We may increase the length of the chain (1 possibility). (c) We may add a point which is incomparable to some other points. Let I be the set of points, to which the new point is incomparable. I is not empty, containing at least the root. It is not the full set of points. And it is an interval. There are in total $\binom{n+1}{2} - 1$ possibilities. ☐

Lemma 4.13. *The fork $\mathfrak{Fk}_n$ has order cocovering number 5.*

Proof. We may increase a cluster by a point. This gives only 2 possibilities, since increasing any of the final clusters gives the same result modulo isomorphism. We may add a point of depth 2, or a point of depth 0. The last option gives 2 possibilities. For a new point of depth 0 may be a successor of a point of depth 0 in $\mathfrak{Fk}_k$ (and therefore only one such point), or not, in which case we get the frame $\mathfrak{Fk}_{n+1}$. ☐

FIGURE 3. Distinguishing the forks and the tacks



Lemma 4.14. *The kite $\mathfrak{Fk}_n \otimes \circ$ has order cocovering number $n + 5$.*

Proof. We may increase one of the clusters (3 possibilities), add a point of depth 3 (1 possibility), of depth 1 (1 possibility) or of depth 0. In the last case, the point is seen by a subset of the set of points of depth 1. This set can have any cardinality. Hence n possibilities arise. ☐

These facts show already that the chains cannot be mapped onto the kites, since the cocovering numbers are distinct. Therefore, we can identify in the lattice NExt **S4** the clusters, the kites and the chains. As regards the tacks and the forks, we have still not succeeded. So, take the tack of three elements, $\mathfrak{T} := \mathfrak{Cl}_2 \otimes \circ$, and the fork $\mathfrak{F} := \mathfrak{Fk}_2$. (To understand the argumentation, it is helpful to look at Figure 3.) Both have three points, and their cocovering number is 5. For sake of contradiction we assume that there is an automorphism α of NExt **S4** that maps $\mathfrak{T}$ onto $\mathfrak{F}$. We know already that it is an involution on NExt **Hd**, and so $\alpha(\mathfrak{F}) = \mathfrak{T}$. Also, $\alpha(\mathfrak{Cl}_2) = \mathfrak{Cl}_2$. We compute the cocovers of $\mathfrak{T}$ and $\mathfrak{F}$ that are below $\mathfrak{Cl}_2$. $\mathfrak{T}$ has exactly one cocover that is below $\mathfrak{Cl}_2$, namely $\mathfrak{P} := \mathfrak{Cl}_2 \otimes \mathfrak{Cl}_2$. $\mathfrak{F}$ has exactly one cocover that is below $\mathfrak{Cl}_2$, which we denote by $\mathfrak{Q}$. Notice that $\mathfrak{P} \not\leq \mathfrak{F}$ and $\mathfrak{Q} \not\leq \mathfrak{T}$. The uniqueness of these elements implies that $\alpha(\mathfrak{P}) = \mathfrak{Q}$ and $\alpha(\mathfrak{Q}) = \mathfrak{P}$. We can also identify the cocovers of $\mathfrak{T}$ and $\mathfrak{F}$ that are handles. They are unique, and they are $\mathfrak{Cl}_3 \otimes \circ$ for $\mathfrak{T}$ and $\mathfrak{Fk}_3$ for $\mathfrak{F}$. Again, these two elemtns are exchanged by α. Now we look at cocovers of $\mathfrak{P}$ that are *not* below $\mathfrak{Cl}_3$, $\mathfrak{Cl}_3 \otimes \circ$ or $\mathfrak{F}$. $\mathfrak{P}$ has no such cocover. We must expect therefore that $\mathfrak{Q} = \alpha(\mathfrak{P})$ has no cocover that is *not* below $\alpha(\mathfrak{Cl}_3)$, $\alpha(\mathfrak{Cl}_3 \otimes \circ)$ or $\alpha(\mathfrak{F})$, that is, not below $\mathfrak{Cl}_3$, $\mathfrak{Fk}_3$ or $\mathfrak{T}$. However, $\mathfrak{Q}$ has such a cocover, $\mathfrak{R}$. This is the desired contradiction. Hence we see that the three element tack and the three element fork may not be interchanged by an automorphism. This is all we need to know for

Proposition 4.15. *Let α be an automorphism of NExt **S4**. Then α fixes the handles pointwise.*

This result can be strengthened in many ways, for example to the lattice of all **S4**–logics of finite codimension. Further, by inspection of the cocovering numbers one can show

Proposition 4.16. *Let α be an automorphism of NExt **Grz**. Then α fixes the handles pointwise.*

5. LATTICE DEFINABLE PROPERTIES OF FRAMES

We will draw some immediate consequences from the previous theorems. Before we do so, however, we will outline the basic philosophy behind the proofs. Given a frame $\mathfrak{F}$, it is rather straightforward to compute its lattice of extensions; it is moreover easy to determine how $\mathsf{Th}\,\mathfrak{F}$ is related to $\mathsf{Th}\,\mathfrak{G}$ for some $\mathfrak{G}$. (If we want to compute these answers, we must assume here that $\mathfrak{F}$ and $\mathfrak{G}$ are finite or in some sense 'effective'.) Finally, given $\mathfrak{F}$, we can determine its position in the lattice $\mathcal{E}\,\Lambda$ for any Λ rather straightforwardly. If we do not know the underyling frame, however, the problem is by far more difficult. It is related with our question about automorphisms in the following way. Suppose that NExt Λ has a nontrivial automorphism α, and let $\Theta \supseteq \Lambda$. Then $\alpha(\Theta)$ and Θ cannot be distinguished by inspection of the lattice NExt Λ. On the other hand, if NExt Λ has no nontrivial automorphisms, then every logic can be determined uniquely by the way it is embedded in the lattice. We have established — for example — that an extension of **S4** is the logic of a handle iff it is of finite codimension and its lattice of extensions is linear. We say therefore that the property of being a handle is *lattice–definable* or simply *l–definable* in NExt **S4**.

Definition 5.1. *Let $\mathcal{P}$ be a property of logics. $\mathcal{P}$ is* ***lattice–definable*** *in NExt Λ, or* ***l–definable*** *for short, if for each $\Theta \in$ NExt Λ and each automorphism α of NExt Λ, Θ has $\mathcal{P}$ iff $\alpha(\Theta)$ has $\mathcal{P}$.*

Definition 5.2. *Let $\mathcal{P}$ be a property of frames. $\mathcal{P}$ is* ***lattice–definable*** *or* ***l–definable*** *in NExt Λ if (1) if $\mathfrak{F}$ has $\mathcal{P}$ and $\mathsf{Th}\,\mathfrak{F} = \mathsf{Th}\,\mathfrak{G}$ then $\mathfrak{G}$ also has $\mathcal{P}$, (2) the set of all $\mathsf{Th}\,\mathfrak{F}$ such that $\mathfrak{F}$ has $\mathcal{P}$ is closed under all automorphisms of NExt Θ.*

Lattice definability is usually sufficient for our purposes, but we will often make use of a stronger property than this one, namely *lattice–constructibility* or *l–constructibility*. The definition we are giving below is a little bit vague, since we need to specifiy what we mean by *finite information*. But this will become clear in Definition 5.4.

Definition 5.3. *Let $\mathcal{P}$ be a property of frames. $\mathcal{P}$ is* ***lattice–constructible*** *or* ***l–constructible*** *in NExt Θ if there exists an algorithm which computes whether $\mathfrak{F}$ has $\mathcal{P}$ on the basis of some finite information concerning $\mathsf{Th}\,\mathfrak{F}$.*

This definition is general enough to encompass also the case of infinite frames of even general frames. But this is too general for the present purposes. Since we are dealing only with finite frames, we might as well restrict them to frames defined over the natural numbers, that is, to frames $\langle F, \lhd \rangle$, where $F \subset \omega$ is finite. Then we have a set of frames; and this set is countable. Hence we can restate the definition above, generalizing it at the same time to arbitrary n–ary relations. Moreover, we now take advantage of the fact that *finiteness* is l–constructible in NExt Λ for all transitive Λ. Therefore, the set of logics of finite codimension in NExt Θ, denoted here by NExt Θ^δ, is countable. (Note that NExt Θ^δ is always a lattice, though not necessarily a locale. It may also fail to have a lowest element.)

Definition 5.4. *Let R be an n–ary relation of finite rooted Kripke–frames and $Q := \mathsf{Th}\,[R]$ its direct image under $\mathsf{Th}\,(-)$. R is* ***l–definable*** *in NExt Λ if (1) $R = \mathsf{Th}^{-1}[Q]$ and (2) Q is closed under all automorphisms of NExt Λ. R is* ***l–constructible*** *in NExt Λ if there is a computable function $f : (\mathrm{NExt}\,\Lambda^\delta)^n \to \{0, 1\}$ such that $f(\langle x_i : i < n\rangle) = 1$ iff $\langle x_i : i < n\rangle \in Q$.*

This definition is extended to functions from $(\mathrm{NExt}\,\Lambda^\delta)^n$ to some given set M. We only need the case where $n = 1$. Also, a unary function f defined on the set of finite frames with values in M is called **l–constructible** if there is a computable function $g : \mathrm{NExt}\,\Lambda^\delta \to M$ such that $f = g \circ \mathsf{Th}$. Usually, $M = \omega$, the set of natural numbers, (the cardinality of $\mathfrak{F}$, for example). To continue our example, the property *being a handle* is l–constructible in all lattices NExt Λ where $\Lambda \supseteq$ **S4**, by Theorem 4.8. Moreover, the cardinality of a frame $\mathfrak{F}$ equals the order codimension of its theory. The latter in turn depends only on the structure of NExt $\mathsf{Th}\,\mathfrak{F}$, which can be constructed in finite time from $\mathfrak{F}$. Hence we conclude the following theorem.

Lemma 5.5. *The cardinality of a frame is an l–constructible function in NExt* ***S4****.*

Indeed, say that a property $\mathcal{P}$ of frames is *intrinsically l–definable (intrinsically l–constructible)* if $\mathcal{P}$ is l–definable (l–constructible) and $\mathcal{P}$ depends only on NExt $\mathsf{Th}\,\mathfrak{F}$, that is, if NExt $\mathsf{Th}\,\mathfrak{F} \cong$ NExt $\mathsf{Th}\,\mathfrak{G}$ then $\mathcal{P}(\mathfrak{F})$ iff $\mathcal{P}(\mathfrak{G})$. Likewise define l–constructibility of relations and functions. What we have shown is that the cardinality is an intrinsically l–constructible function.

Furthermore, the type of the handle is also l–constructible. Since each handle is fixed by its cardinality and its type (fork, cluster, etc.) we know that the property *being isomorphic to* $\mathfrak{F}$, where $\mathfrak{F}$ is a handle, is l–constructible. This is in fact our starting base. By means of these results we will establish more and more properties of frames to be l–constructible. In the end, we will have that for every finite rooted **S4**–frame $\mathfrak{F}$ the property of *being isomorphic to* $\mathfrak{F}$ is l–constructible, and this establishes that the lattice NExt $\mathbf{S4}^\delta$ has only one automorphism. In fact, to establish this we show how $\mathfrak{F}$ can be constructed up to isomorphism from $\mathsf{Th}\,\mathfrak{F}$.

We should issue a warning here that l–definability and l–constructibility are relative to the lattice NExt Λ. It may very well be that a property of frames is l–definable in NExt Λ but not in NExt Θ. This may have two reasons. (1) Λ properly extends Θ, but Λ is not fixed under all automorphisms of NExt Θ, (2) Λ properly contains Θ, but NExt Θ admits automorphisms which do not extend to automorphisms of NExt Λ. The second case appears for example with respect to **S4** and **S4.3**. Nevertheless, we will establish many results only for NExt **S4**. The generalizations to arbitary lattices of **S4**–logics are often easy to make, and to state the theorems in their most general form would make them rather unrevealing.

Call a logic of **fatness** k if it is complete with respect to frames of fatness $\leq k$. Equivalently, a logic Λ is of fatness k iff $\Lambda \supseteq \mathbf{S4}/\{\mathfrak{Cl}_{k+1}, \mathfrak{Cl}_{k+1} \otimes \circ\}$. We denote the logic of frames of fatness $\leq k$ by $\mathbf{S4}.f_k$. A particular case is $k = 1$. The logic of frames of fatness 1, $\mathbf{S4}.f_1$, is exactly **Grz**. From Proposition 4.15 and Lemma 3.4 we deduce

Corollary 5.6. *Every automorphism of NExt* **S4** *fixes each logic* $\mathbf{S4}.f_k$*, in particular* **Grz**.

Hence, if $\mathfrak{F}$ is a rooted frame, $\alpha(\mathfrak{F})$ has the same fatness as $\mathfrak{F}$. Furthermore, we deduce that any automorphism of NExt **S4** must induce an automorphism on NExt **Grz**, and this helps in reducing the choices for automorphisms of NExt **S4**.

Given a frame $\mathfrak{F}$, write $\mho^{(k)}(\mathfrak{F})$ for the frame resulting from $\mathfrak{F}$ by reducing all clusters to size $\leq k$. That is to say, if a cluster of $\mathfrak{F}$ has size $\leq k$, it remains untouched, otherwise it is reduced to size k. We call $\mho^{(k)}(\mathfrak{F})$ the k–**skeleton** of $\mathfrak{F}$. For $k = 1$ we speak of the **skeleton** rather than the 1–skeleton. The construction of passing to the k–skeleton can be defined on logics as follows. We put

$$\mho^{(k)}\Lambda := \Lambda \sqcup \mathbf{S4}.f_k$$

It is not hard to see that this does the job. $\mathbf{S4}.f_k$ is fixed by any automorphism of NExt **S4**.

Proposition 5.7. *The functions* $\mho^{(k)}(-)$ *are l-constructible in NExt* **S4**.

Clearly, $\mho^{(k)}\,\mathsf{Th}\,\mathfrak{F} = \mathsf{Th}\,\mho^{(k)}\,\mathfrak{F}$. Hence, *having the same k–skeleton* is an l–definable relation between frames (or logics). Unfortunately, it is not easy to deduce the structure of the k–skeleton of the frame generating a logic. Indeed, this is the main task we have to set ourselves in order to show that all finite rooted frames are fixed by an automorphism of NExt **S4**.

Proposition 5.8. *The function ft, assigning to each* **S4***–frame its fatness, is l–constructible in NExt* **S4**.

Proof. Let $\Theta = \mathsf{Th}\,\mathfrak{F}$. Then the fatness is less or equal to the cardinality of $\mathfrak{F}$. Now, for $k \leq \sharp\mathfrak{F}$ check whether $\Theta = \mho^{(k)}\Theta$. This can be done in finite time. Since for $k = \sharp\mathfrak{F}$ we have equality, the exists a smallest k for which $\Theta = \mho^{(k)}\Theta$. This k is the fatness of $\mathfrak{F}$. □

We have previously seen that the cardinality of a finite rooted frame is l–constructible. Now, let $\gamma_k(\mathfrak{F}) := \sharp\mho^{(k)}\,\mathfrak{F} - \sharp\mho^{(k-1)}\,\mathfrak{F}$, $k > 1$, and $\gamma_1(\mathfrak{F}) := \sharp\mho^{(1)}\,\mathfrak{F}$. Clearly, $\gamma_k(\mathfrak{F})$ is the number of clusters of size k in $\mathfrak{F}$.

Lemma 5.9. *The number of clusters of size k is invariant under all automorphisms of NExt* **S4**.

Proof. Let Λ be a logic of finite codimension. $\alpha(\mho^{(k)}\Lambda) = \alpha(\Lambda \sqcup \mathbf{S4}.f_k) = \alpha(\Lambda) \sqcup \alpha(\mathbf{S4}.f_k)$. Since $\mathbf{S4}.f_k \in Fix(\alpha)$, we get $\alpha(\mho^{(k)}\Lambda) = \mho^{(k)}(\alpha(\Lambda))$. The order codimension of a logic is invariant under any automorphism. Hence $\mho^{(k)}\Lambda$ and $\alpha(\mho^{(k)}\Lambda)$ have the same order codimension. It follows that their generating frames have the same number of points. So, the number of clusters of a given size is invariant under any automorphism. □

We can restate this theorem in another, perhaps more visual way.

Definition 5.10. *Let $\mathfrak{F}$ be an **S4**–frame. Then* bw$(\mathfrak{F})$ *denotes the multiset of all* $\sharp C$*, where* C *is a nonfinal cluster.* bw$(\mathfrak{F})$ *is called the* ***body weight*** *of* $\mathfrak{F}$*.* tw$(\mathfrak{F})$*, the* ***tail weight*** *of* $\mathfrak{F}$*, is the multiset of all* $\sharp C$*, where* C *is a final cluster. Finally, the* ***weight*** *of* $\mathfrak{F}$*,* wt$(\mathfrak{F})$*, is the multiset union of the body weight and the tail weight. Equivalently, it is the multiset of all* $\sharp C$*, where* C *is a cluster of* $\mathfrak{F}$*.*

For example, the body weight of the tack $\mathfrak{Cl}_k \otimes \circ$ is $\{k\}_m$, its tail weight is $\{1\}_m$, and the weight is $\{1, k\}_m$. The subscript $_m$ reminds us that we are speaking of multisets, not of sets. The chain $\mathfrak{Ch}_4$ has body weight $\{1, 1, 1\}_m$, tail weight $\{1\}_m$, and its weight is $\{1, 1, 1, 1\}_m$. Notice that the multiset union, intersection and difference (denoted by $\cup_m$, $\cap_m$ and $-_m$, respectively) take notice of the multiplicities of elements. If A contains an element x p times and B contains x q times then $A \cup_m B$ contains x $p+q$ times, $A \cap_m B$ contains x $min\{p, q\}$ times, and $A -_m B$ contains x exactly $p - q$ times if $p \geq q$, and 0 times else. The three weight functions are connected with each other as follows.

$$\begin{array}{rcl} wt(\mathfrak{F}) & = & bw(\mathfrak{F}) \cup_m tw(\mathfrak{F}) \\ bw(\mathfrak{F}) & = & wt(\mathfrak{F}) -_m tw(\mathfrak{F}) \\ tw(\mathfrak{F}) & = & wt(\mathfrak{F}) -_m bw(\mathfrak{F}) \end{array}$$

Given the numbers $\gamma_k(\mathfrak{F})$, the weight of $\mathfrak{F}$ is the multiset containing the number k exactly $\gamma_k(\mathfrak{F})$ times, for each k. (Clearly, if k exceeds the fatness of $\mathfrak{F}$, $\gamma_k(\mathfrak{F}) = 0$, and so nothing is added to the multiset.) The following theorem is a restatement of Lemma 5.9 with respect to the weights of $\mathfrak{F}$.

Lemma 5.11. *Let α be an automorphism of NExt* ***S4***. *Then* bw$(\alpha(\mathfrak{F}))$ = bw$(\mathfrak{F})$, tw$(\alpha(\mathfrak{F}))$ = tw$(\mathfrak{F})$ *and* wt$(\alpha(\mathfrak{F}))$ = wt$(\mathfrak{F})$. *In other words, the body weight, the tail weight and the weight are invariant under all automorphisms of NExt* ***S4***. *Moreover, the weight functions are l–constructible in NExt* ***S4***.

Proof. Let $\delta_k(\mathfrak{F})$ be the number of final clusters of $\mathfrak{F}$ of size k. We show that this number is invariant. To this end, we define the operation $F_k : \Lambda \mapsto \Lambda \sqcup \mathbf{S4}/\mathfrak{Cl}_{k+1}$. Its effect on the generating frame is to reduce the final clusters of size $> k$ to clusters of size k. Now reason as in Lemma 5.9. To show the theorem for the body weight, we appeal to the fact that the body weight is the multiset–difference of the weight and the tail weight. Alternatively, we can define the function $B_k : \Lambda \mapsto \Lambda \sqcup \mathbf{S4}/\mathfrak{Cl}_{k+1} \otimes \circ$ and reason in the same way as before. ☐

A logic containing **S4** is said to be of **depth** n if it is complete with respect to frames of depth $\leq n$. The logic of **S4**–frames of depth $\leq n$ is called $\mathbf{S4}_n$. It is the logic of all **S4**–frames of depth $\leq n$. $\mathbf{S4}_n$ is the result of splitting a handle from **S4**, namely the chain $\mathfrak{Ch}_{n+1}$.

Corollary 5.12. *Every automorphism of NExt* ***S4*** *fixes* $\boldsymbol{S4}_n$ *for all* n.

So, the depth of a frame is also invariant under automorphisms. The depth function is l–constructible in NExt **S4**, as can be seen easily. Finally, let us note that $\mathbf{S4.3} = \mathbf{S4}/\{\mathfrak{Fk}_2, \mathfrak{Fk}_2 \otimes \circ\}$ (see [8]).

Corollary 5.13. *Every automorphism of NExt* ***S4*** *fixes* ***S4.3***.

This allows us to deduce various important results on automorphisms of NExt **S4** from results on $\mathfrak{Aut}$(NExt **S4.3**). It follows that an automorphism of NExt **S4** fixes NExt **S4.3**, though we cannot conclude that it fixes the lattice pointwise. However, this is the case, as we will show in Theorem 7.1.

6. The Group of Automorphisms of NExt S4.3

NExt **S4.3** is continuous (see [6]). A logic is prime iff it is the logic of a finite rooted frame. Therefore, by the results of §2, any automorphism of the poset of finite rooted frames (up to isomorphism) induces an automorphism of NExt **S4.3**. We will therefore study possible automorphisms of this structure. We write $\langle k_i : i < n\rangle$ for the frame of depth n whose cluster of depth j contains k_j elements. Obviously, $k_i > 0$ for all i. Given $\mathfrak{F}$ let $\sigma(\mathfrak{F})$ denote the sequence $\langle k_i : i < n\rangle$ where k_j is the cluster size of the cluster of depth j of $\mathfrak{F}$. For example, $\sigma(\mathfrak{Cl}_n) = \langle n\rangle$, $\sigma(\mathfrak{Cl}_n \otimes \circ) = \langle 1, n\rangle$ and $\sigma(\mathfrak{Ch}_3) = \langle 1, 1, 1\rangle$. Let $\gamma = \langle k_i : i < n\rangle$ and $\delta = \langle m_j : j < p\rangle$. Write $\gamma \leq \delta$ if there is a strictly ascending sequence $j(i)$, $i < p$, such that $j(0) = 0$ and $m_{j(i)} \leq k_i$ for all $i < p$. It follows that $\mathsf{Th}\,\mathfrak{F} \subseteq \mathsf{Th}\,\mathfrak{G}$ iff $\sigma(\mathfrak{F}) \leq \sigma(\mathfrak{G})$. Hence, we may restrict ourselves to the study of the automorphisms of the order $\langle(\omega - \{0\})^+, \leq\rangle$, where $(\omega - \{0\})^+$ is the set of finite, nonempty sequences of nonzero numbers.

The linear handles are fixed under any automorphism of **S4.3**. This does not follow from the previous results but can be established in the same way. First of all, it follows that the set of handles is fixed, though not necessarily pointwise. There are only three types of handles: the clusters, the tacks and the chains. The cocovering numbers are now: 2 for the n point cluster, 3 for the tacks, and $n + 1$ for the n element chain. Since for large enough n these numbers are distinct, it follows that the set of handles is fixed pointwise.

Lemma 6.1. *Every automorphism of NExt **S4.3** fixes the set of handles pointwise.*

An immediate corollary, using Lemma 3.3, is

Lemma 6.2. *Every automorphism of NExt **S4.3** fixes the set of logics **S4**.f_k pointwise.*

Since by Lemma 6.1 the logic of the n–point cluster is invariant under any automorphism, it follows that the logic $\mathbf{S4.3}/\mathfrak{Cl}_n$ is also fixed by any automorphism. Hence $\alpha(\mathfrak{F})$ has the same tail weight as $\mathfrak{F}$.

Lemma 6.3. *Let α be an automorphism of **S4.3** and $\mathfrak{F}$ a rooted frame for* **S4.3**. *Then $\alpha(\mathfrak{F})$ and $\mathfrak{F}$ have the same tail weight and the same body weight.*

This means in effect that the automorphism can only permute the nonfinal clusters of a frame. Now we shall determine the kinds of permutations that are induced by an automorphism α. Let $\mathfrak{x}(n, i, k)$ be a frame of length n with weight $\{k, 1, 1, \ldots\}_m$, where the cluster of depth i has size k. It is easy to see that

$$\langle k_i : i < n\rangle = glb\{\mathfrak{x}(n, i, k_i) : i < n\}$$

(Here, $glb\,M$ denotes the greatest lower bound of M.) We call a frame a **snake** if it is of the form $\mathfrak{x}(n, i, k)$ for some numbers i, k, n. By the results above, α fixes the set of snakes. It follows from the equation above that

Lemma 6.4. *α is an isomorphism of NExt **S4.3** iff it induces an isomorphism on the partial order of the logics of snakes.*

It suffices therefore to study automorphisms α of the partial order of the logics of snakes. α fixes the set of snakes of a given length and a given k. Since any frame is the greatest lower bound of a set of snakes, it is enough to study the action of α on snakes. The next lemma reduces the set to be looked at even more.

Lemma 6.5. *Suppose that $\alpha(\mathfrak{x}(n, i, 2)) = \alpha(\mathfrak{x}(n, j, 2))$. Then $\alpha(\mathfrak{x}(n, i, k)) = \alpha(\mathfrak{x}(n, j, k))$ for any $k \geq 2$.*

Proof. $\mathfrak{x}(n, i, k)$ is uniquely determined by the fact that it is a snake of fatness k and length n, and is below $\mathfrak{x}(n, i, 2)$. Since α leaves length and fatness invariant, it follows that $\alpha(\mathfrak{x}(n, i, k))$ is a snake of fatness k, length n, and below $\alpha(\mathfrak{x}(n, i, 2)) = \mathfrak{x}(n, j, 2)$. Hence $\alpha(\mathfrak{x}(n, i, k)) = \mathfrak{x}(n, j, k)$. ☐

α fixes the set $\{\mathfrak{x}(3, 1, 2), \mathfrak{x}(3, 2, 2)\} = \{\langle 1, 2, 1\rangle, \langle 1, 1, 2\rangle\}$. Let us assume that $\alpha(\langle 1, 1, 2\rangle) = \langle 1, 1, 2\rangle$. It follows that $\alpha(\mathfrak{x}(n, n-1, 2)) = \mathfrak{x}(n, n-1, 2)$, $n > 2$. Namely, $\langle 1, 2, 1\rangle \not\geq \mathfrak{x}(n, n-1, 2)$ and so $\langle 1, 2, 1\rangle = \alpha(\langle 1, 2, 1\rangle) \not\geq \alpha(\mathfrak{x}(n, n-1, 2))$. Hence $\alpha(\mathfrak{x}(n, n-1, 2)) = \mathfrak{x}(n, n-1, 2)$. Also, $\alpha(\mathfrak{x}(n, 1, 2)) = \mathfrak{x}(n, 1, 2)$, by an analogous argument. Let us now assume that $\alpha(\langle 1, 1, 2\rangle) = \langle 1, 2, 1\rangle$. Then by the same argument, $\alpha(\mathfrak{x}(n, n-1, 2)) = \mathfrak{x}(n, 1, 2)$ and $\mathfrak{x}(n, 1, 2) = \mathfrak{x}(n, n-1, 2)$.

Lemma 6.6. *Suppose that $\alpha(\langle 1,1,2\rangle) = \langle 1,1,2\rangle$. Then α is the identity.*

Proof. By induction on the length of the snakes we show that $\alpha(\mathfrak{x}(n,i,2)) = \mathfrak{x}(n,i,2)$. The case $n = 3$ is settled. Assume that α is the identity on all snakes of length $\leq n$ where $n \geq 3$. α is a permutation of the set $\{\mathfrak{x}(n+1,i,2) : 0 < i < n+1\}$. We have

$$\mathfrak{x}(n,i,2) \geq \mathfrak{x}(n+1,j,2) \text{ iff } j = i \text{ or } j = i+1$$

Therefore,

$$\alpha(\mathfrak{x}(n,i,2)) \geq \alpha(\mathfrak{x}(n+1,j,2)) \text{ iff } j = i \text{ or } j = i+1$$

We have shown that $\alpha(\mathfrak{x}(n+1,n,2)) = \mathfrak{x}(n+1,n,2)$. Now assume that $\alpha(\mathfrak{x}(n+1,i+1,2)) = \mathfrak{x}(n+1,i+1,2)$. By assumption on n this gives

$$\mathfrak{x}(n,i,2) \geq \alpha(\mathfrak{x}(n+1,i,2)), \alpha(\mathfrak{x}(n+1,i+1,2)$$

Therefore $\{\alpha(\mathfrak{x}(n+1,i,2), \alpha(\mathfrak{x}(n+1,i+1,2))\} = \{\mathfrak{x}(n+1,i,2), \mathfrak{x}(n+1,i+1,2)\}$. It follows that $\alpha(\mathfrak{x}(n+1,i,2)) = \mathfrak{x}(n+1,i,2)$. So, α must fix all $\mathfrak{x}(n+1,i,2)$. This establishes the claim for $n+1$. ☐

Lemma 6.7. *Suppose that $\alpha(\langle 1,1,2\rangle) = \langle 1,2,1\rangle$. Then $\alpha(\mathfrak{x}(n,i,2)) = \mathfrak{x}(n,n-i,2)$ for all n and i.*

Proof. A similar argument. Assume that $\alpha(\mathfrak{x}(n,i,2)) = \mathfrak{x}(n,n-i,2)$ for all $0 < i < n$. We show that then $\alpha(\mathfrak{x}(n+1,i,2)) = \mathfrak{x}(n+1,n+1-i,2)$ for all $0 < i < n+1$. The claim then follows, since for $n = 3$ it holds by assumption on α. Assume that $\alpha(\mathfrak{x}(n+1,i+1,2)) = \mathfrak{x}(n+1,n-i,2)$. We aim to show that $\alpha(\mathfrak{x}(n+1,i,2)) = \alpha(n+1,n+1-i,2)$. Since $\alpha(\mathfrak{x}(n+1,1,2)) = \mathfrak{x}(n+1,n,2)$, the claim is then established. Recall that

$$\mathfrak{x}(n,i,2) \geq \mathfrak{x}(n+1,j,2) \text{ iff } j = i \text{ or } j = i+1$$

Hence

$$\alpha(\mathfrak{x}(n,i,2)) \geq \alpha(\mathfrak{x}(n+1,j,2)) \text{ iff } j = i \text{ or } j = i+1$$

By induction hypothesis this gives

$$\mathfrak{x}(n,n-i,2)) \geq \alpha(\mathfrak{x}(n+1,j,2)) \text{ iff } j = i \text{ or } j = i+1$$

Hence $\alpha(\mathfrak{x}(n+1,j,2)) \in \{\mathfrak{x}(n+1,n-i,2), \mathfrak{x}(n+1,n+1-i,2)\}$. By inductive hypothesis, $\alpha(\mathfrak{x}(n+1,i+1,2) = \mathfrak{x}(n+1,n+1-i,2)$. Therefore $\alpha(\mathfrak{x}(n+1,i,2)) = \mathfrak{x}(n+1,n-i,2)$. ☐

Let $\mathbb{Z}_n$ denote the cyclic group of order n. Then we have the following main result.

Theorem 6.8. $\mathfrak{Aut}(\mathit{NExt}\,\boldsymbol{S4.3}) \cong \mathbb{Z}_2$.

Although the proof is now complete by Lemma 6.4, we will spell out the details more concretely. α is a permutation of $S_3 = \{\langle 1,1,2\rangle, \langle 1,2,1\rangle\}$. By Lemma 6.6, if α is the identity on S_3, α is the identity on the snakes of fatness 2, and by Lemma 6.5 it is the identity on all snakes. This implies that α is the identity. Now let $\alpha(\langle 1,1,2\rangle) = \langle 1,2,1\rangle$. By Lemma 6.7 and Lemma 6.5, $\alpha(\mathfrak{x}(n,i,k)) = \mathfrak{x}(n,n-i,k)$ for all snakes. From this we deduce that $\alpha(\langle k_i : i < n\rangle) = \langle k_0, k_{n-1}, k_{n-2}, \dots, k_2, k_1\rangle$. It remains to be shown that α is an isomorphism of the poset $\langle(\omega - \{0\})^+, \leq\rangle$. Therefore, let $\gamma = \langle m_i : i < p\rangle$ and $\delta = \langle n_j : j < q\rangle$ and assume that $\gamma \leq \delta$. Then there exists a strictly ascending sequence $\langle s(j) : i < q\rangle$ such that $j(0) = 0$ and $m_{s(j)} \geq n_j$ for all $j < q$. We have to show that $\alpha(\gamma) \leq \alpha(\delta)$.

$$\begin{array}{rclcl} \alpha(\gamma) & = & \langle m'_i : i < p\rangle & = & \langle m_0, m_{p-1}, m_{p-2}, \dots, m_2, m_1\rangle \\ \alpha(\delta) & = & \langle n'_j : j < q\rangle & = & \langle n_0, n_{q-1}, n_{q-2}, \dots, n_2, n_1\rangle \end{array}$$

The sequence $t(j)$ defined by $t(0) := s(0)$ and $t(j) := p - s(q-i)$ is strictly ascending as well. Moreover, $n'_{t(0)} = n_0 \geq m_0 = m'_0$ and for $0 < j < q$ we have $n'_{t(j)} = n_{s(q-j)} \geq m_{q-j} = m'_j$. Therefore, $\alpha(\gamma) \leq \alpha(\delta)$. Since α is an involution, it follows from $\alpha(\gamma) \leq \alpha(\delta)$ that $\gamma \leq \delta$. Hence, α is an isomorphism.

It follows that the l–indeterminacy of logics with respect to **S4.3** is either 1 or 2. This in turn means that not all logics are uniquely determined by their position in the lattice NExt **S4.3**. We remark here that there is a rather fast intuitive proof of Lemma 6.6 and 6.7. Namely, the poset of snakes of the form $\mathfrak{x}(n, i, 2)$, $n, i > 0$, is isomorphic to the set ω^2 ordered by $\langle i, j\rangle \leq \langle i', j'\rangle$ iff $i \geq i'$ and $j \geq j'$, which in turn is isomorphic to the poset underlying the lattice $\omega^{\perp} \times \omega^{\perp}$, where $\omega^{\perp} = \langle \omega, \geq\rangle$. It is not hard to see that this poset has exactly two automorphisms.

7. The Automorphisms of NExt S4

We have seen in the previous section that there are only two automorphisms of NExt **S4.3**. Here we will attack the question of automorpshims of NExt **S4**. We already know that every automorphism of NExt **S4** fixes **S4.3**. Hence it induces on NExt **S4.3** an automorphism. We will show that this automorphism is always the identity. Hence only the identity on NExt **S4.3** can be extended to an automorphism of NExt **S4**, though the extension need not be the identity itself.

Theorem 7.1. *Every automorphism of NExt* ***S4*** *fixes NExt* ***S4.3*** *pointwise.*

Proof. Consider the frame to the left.

Call this frame $\mathfrak{F}$. $\mathfrak{F}$ is obtained form a kite by blowing up the middle clusters. This frame is below the snake $\mathfrak{x}(3, 1, 2)$ but not below $\mathfrak{x}(3, 2, 2)$. Consider $\alpha(\mathfrak{F})$. This frame is either below $\mathfrak{x}(3, 1, 2)$ or below $\mathfrak{x}(3, 2, 2)$, but not both. If $\alpha(\mathfrak{F})$ is below $\mathfrak{x}(3, 2, 2)$, then it is not below $\mathfrak{x}(3, 1, 2)$. $\alpha(\mathfrak{F})$ has fatness 2, and is the frame shown to the right. This frame has cardinality 5. But $\mathfrak{F}$ has cardinality 6. Contradiction. Hence $\alpha(\mathfrak{F})$ is below $\mathfrak{x}(3, 1, 2)$. It follows that $\alpha(\mathfrak{x}(3, 1, 2)) = \mathfrak{x}(3, 1, 2)$, and so α is the identity. ☐

This gives us a good start. Unfortunately, the lattice of normal extensions of **S4** is far more complicated than the lattice of extensions of **S4.3**. For unlike **S4.3**, not all extensions of **S4** have the finite model property, and so the action on the logics of finite codimensions may not be enough to determine the action of the automorphism on the entire lattice. Nevertheless, it is already a rather intricate problem to show that any automorphism must fix the elements of finite codimension pointwise. This is what we will prove now, leaving the full problem unsolved for the moment.

We can sharpen Lemma 5.11 as follows. Let $\mathbf{S4}.f(k, d)$ be the logic of frames whose cluster of depth $\geq d$ have fatness k. Then $\mathbf{S4}.f_k = \mathbf{S4}.f(k, 0)$. It turns out that for $d > 0$,

$$\mathbf{S4}.f(n, d) = \mathbf{S4}/\mathfrak{x}(d + 1, k + 1, d)$$

For suppose that $\mathfrak{F}$ has a cluster C of size $> k$ of depth $d > 0$. Then let $\mathfrak{G}$ be the subframe generated by C. $\mathfrak{G}$ can be mapped onto the snake $\mathfrak{x}(d + 1, k + 1, d)$. Conversely, if there exists a subframe of $\mathfrak{F}$ that can be mapped onto the snake $\mathfrak{x}(d + 1, k + 1, d)$, then $\mathfrak{F}$ contains a cluster of size at least $k + 1$ which is of depth at least d.

Definition 7.2. *Let $\mathfrak{F}$ be a finite* ***S4****–frame. Then $\gamma_k(\mathfrak{F}, d)$ denotes the number of clusters of size k at depth d in $\mathfrak{F}$.*

So, $\gamma_k(\mathfrak{F}) = \sum_{d\in\omega} \gamma_k(\mathfrak{F}, d)$. Now let $\mho^{(k:d)}(\mathfrak{F})$ be the generating frame of $\mathsf{Th}\,\mathfrak{F} \sqcup \mathbf{S4}.f(d,k)$. (In other words, we squash all clusters of size $> k$ of depth $\geq d$ onto clusters of size k.) Consider the number $\beta_k(\mathfrak{F}, d) := \sharp f - \sharp\mho^{(k:d)}\mathfrak{F}$. Suppose that $\mathfrak{F}$ is of fatness $k+1$. Then $\beta_k(\mathfrak{F}, d)$ counts exactly the number of clusters of size $k+1$ of depth $\geq d$. Hence, the number of clusters of size $k+1$ and depth $= d$ can be computed for all d. Now we continue this procedure with $\mho^{(k:0)}\mathfrak{F}$ in place of $\mathfrak{F}$ and thereby determine the number of clusters of size $= k-1$ of given depth of $\mho^{(k:0)}\mathfrak{F}$. This is however the same as the number of clusters of size $\geq k-1$ of given depth of $\mathfrak{F}$. And so forth. The following is now immediate.

Lemma 7.3. *Let α be an automorphism of NExt* ***S4****. Then for all finite rooted* ***S4****–frames $\mathfrak{F}$ and natural numbers d: $\gamma_k(\alpha(\mathfrak{F}), d) = \gamma_k(\mathfrak{F}, d)$.*

Let $\mathfrak{F}$ be a frame. Then $\mathfrak{Aut}(\mathfrak{F})$ denotes the group of automorphisms of $\mathfrak{F}$. Let x be a point of $\mathfrak{F}$. Then denote by $[x]$ the orbit of x under $\mathfrak{Aut}(\mathfrak{F})$. We denote by $\partial\mathfrak{F}$ the following frame. Its set of worlds is $\{[x] : x \in f\}$ and we put $[x] \lhd [y]$ iff there exists $x' \in [x]$ and $y' \in [y]$ such that $x' \lhd y'$. We call this the **derived frame** of $\mathfrak{F}$. The following holds.

Proposition 7.4. *Let $\mathfrak{F}$ be an* **S4***–frame. Then $\partial\,\mathfrak{F}$ is slender. Moreover, the map $\partial : x \mapsto [x]$ is a p–morphism from $\mathfrak{F}$ onto $\partial\,\mathfrak{F}$.*

Proof. If x and x' are in the same cluster, there is an automorphism mapping x to x'. (In fact, the map which exchanges x and x' and is the identity otherwise is an automorphism.) Hence $\partial\,\mathfrak{F}$ is slender. To show that ∂ is a p–morphism, we need to prove that if $[x] \lhd [y]$ then there exists a $y' \in [y]$ such that $x \lhd y'$. By assumption there exist $\widehat{x} \in [x]$ and $\widehat{y} \in [y]$ such that $\widehat{x} \lhd \widehat{y}$. Furthermore, there exists an automorphism α such that $\alpha(\widehat{x}) = x$, by definition of $[x]$. Put $y' := \alpha(\widehat{y})$. Then $y' \in [y]$, and $x = \alpha(\widehat{x}) \lhd \alpha(\widehat{y}) = y'$, since α is an automorphism. ☐

Definition 7.5. *A frame $\mathfrak{F}$ is called* ***rigid*** *if the identity is the only automorphism of $\mathfrak{F}$.*

It may be thought that $\partial\,\mathfrak{F}$ is a rigid frame for every $\mathfrak{F}$. This is not so. In fact, derivation sequences of frames can assume any finite length.

Lemma 7.6. *For every number n there is a finite frame $\mathfrak{F}$ such that $\partial^{n-1}\,\mathfrak{F} \not\cong \partial^n\,\mathfrak{F}$.*

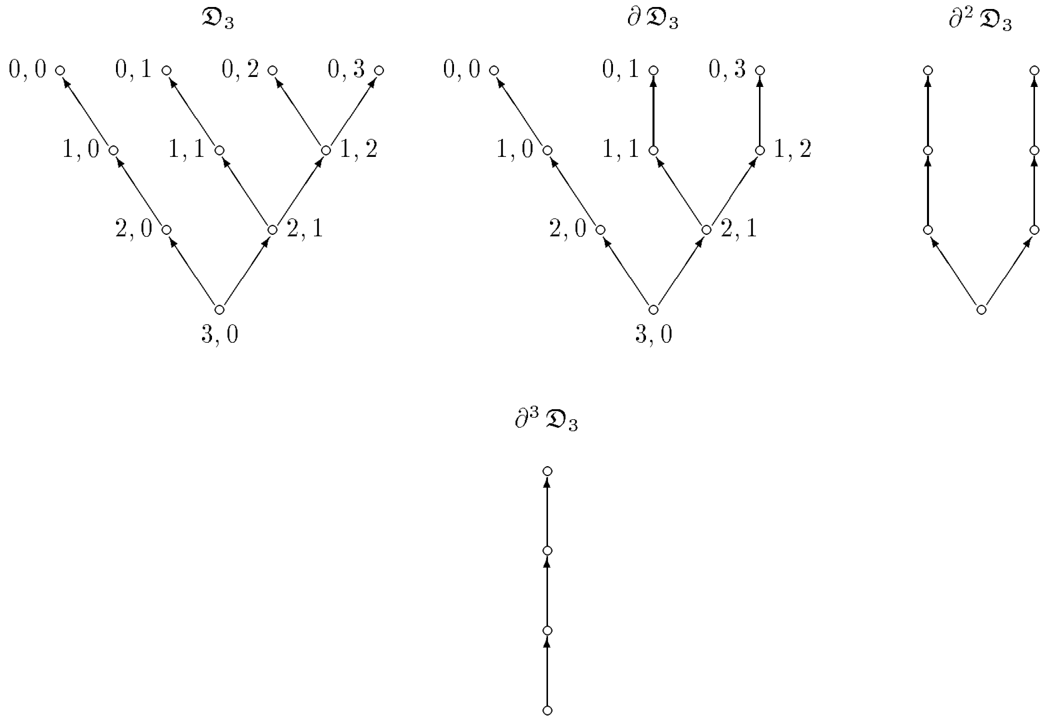
Proof. Take $\mathfrak{D}_n := \langle D_n, \lhd_n\rangle$, where $D_n := \{\langle p, q\rangle : p+q \leq n\}$ and $\langle p, q\rangle \lhd_n \langle p', q'\rangle$ iff (i) $p' \leq p$ and $q' = q$ or (ii) $p + q = p' + q' = n$ and $p' \leq p$. It turns out that $\partial^n\mathfrak{D}_n \cong \mathfrak{Ch}_n$, but $\partial^{n-1}\mathfrak{D}_n \not\cong \mathfrak{Ch}_n$. For a proof the reader may take a look at Figure 4. ☐

We will make heavy use of the skeleton. Suppose that we are given two frames $\mathfrak{F}$ and $\mathfrak{G}$ with identical skeleton such that $\mathfrak{G} \leq \mathfrak{F}$. Then the interval $[\mathsf{Th}\,\mathfrak{G}, \mathsf{Th}\,\mathfrak{F}]$ in $\langle Ir(\mathrm{NExt}\,\mathbf{S4}), \leq\rangle$ (the partial order of irreducible **S4**–logics) is called the **matching space** of $\mathfrak{F}$ and $\mathfrak{G}$ and denoted by $M(\mathfrak{G}, \mathfrak{F})$. The matching space is a partially ordered set. We can define the **codimension** of Λ in $M(\mathfrak{G}, \mathfrak{F})$ to be the maximum size of a maximal properly ascending chain from Λ to $\mathsf{Th}\,\mathfrak{F}$ diminished by 1. (There may be several such maximal chains, so we only look at the length of the longest of them.) The matching space of $\mathfrak{F}$ and $\mathfrak{G}$ consists of all those irreducible logics whose generating frames are rooted frames which have the same skeleton as $\mathfrak{F}$ (and as $\mathfrak{G}$), but the size of their clusters is between that of the corresponding cluster in $\mathfrak{F}$ and the corresponding cluster of $\mathfrak{G}$. The structure of the matching space is not entirely straightforward to construct from $\mathfrak{F}$ and $\mathfrak{G}$. Figure 5 gives an example of a matching space for the fork $\mathfrak{Fk}_3$ and the frame formed by blowing up each cluster to two points.

Definition 7.7. *Let $\mathfrak{F}$ be a frame and C a cluster of $\mathfrak{F}$. Denote by $\boxplus_C\,\mathfrak{F}$ the result of adding a point to C, and by $\boxminus_C\,\mathfrak{F}$ the result of removing a point from C. The map $C \mapsto \mathsf{Th}\ \boxplus_C\ \mathfrak{F}$ is called the* ***trimming map*** *and $C \mapsto \mathsf{Th}\ \boxminus_C\ \mathfrak{F}$ the* ***inverse trimming map****.*

Notice that $\boxplus_C\,\mathfrak{F}$ is defined only up to isomorphism. Further, if C has only one point, the operation $\boxminus_C\,\mathfrak{F}$ effectively removes C. We will however not make use of $\boxminus_C$ in that situation. By definition, $\mathfrak{F}$ is a subframe of $\boxplus_C\,\mathfrak{F}$ and $\boxminus_C\,\mathfrak{F}$ is a subframe of $\mathfrak{F}$. $\boxplus_C$ is therefore iterable, and $\boxplus_C^k\,\mathfrak{F}$

FIGURE 4. The derivation sequence of $\mathfrak{D}_3$



is the result of adding k many points to C, and $\boxminus^k_C\,\mathfrak{F}$ — if defined — is the result of removing k many points from C. Although $\boxminus_C \boxplus_C\ \mathfrak{F}$ is generally not identical to $\mathfrak{F}$ but isomorphic to it, we assume here for simplicity that the two are identical. Similarly with $\boxplus_C \boxminus_C\ \mathfrak{F}$ ($\sharp C > 1$). The following is now clear.

Lemma 7.8. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite* **S4***–frames with isomorphic skeletons and $\mathfrak{G} \leq \mathfrak{F}$. Then $\mathfrak{G}$ is isomorphic to some $\mathfrak{H}$ which is obtained from $\mathfrak{F}$ by a series of operations of the form $\boxplus_C$, C a cluster of $\mathfrak{F}$.*

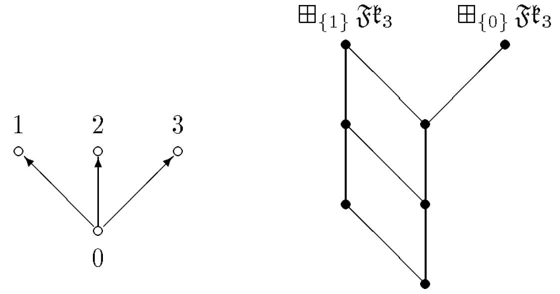
So, in a matching space, we can move from higher elements to lower elements by means of trimming maps. The matching space is a central construction. By embedding an irreducible logic carefully into some (l–constructible) matching space we will be able to extract the structure of the generating frame.

There is a construction dual to $\mho^{(k)}\,\Lambda$, called $\Omega_{(k)}\,\Lambda$. For a frame $\mathfrak{F}$, we denote by $\Omega_{(k)}\,\mathfrak{F}$ the frame $\mathfrak{G}$ with the least number of worlds such that (1) $\mathfrak{G} \leq \mathfrak{F}$, (2) $\mho^{(1)}\,\mathfrak{G} \cong \mho^{(1)}\,\mathfrak{F}$, and (3) $\gamma_j(\mathfrak{G}) = 0$ for each $j < k$. ((3) says that every cluster of $\mathfrak{G}$ must have size $\geq k$.) It is not hard to see that $\mathfrak{G}$ is unique up to isomorphism. Then if $\Lambda = \mathsf{Th}\,\mathfrak{F}$ we put $\Omega_{(k)}\Lambda := \mathsf{Th}\,\Omega_{(k)}\,\mathfrak{F}$. Although this construction is not as easily describable in lattice theoretic terms, it is nevertheless clear that $\Omega_{(k)}\,\Lambda$ is l–constructible in NExt **S4**. Let now $\Theta = \mathsf{Th}\,\mathfrak{F}$ for some rooted frame of fatness k. Then we have the following sequence

$$\ldots \mho_{(k+2)}\,\Theta \leq \mho_{(k+1)}\,\Theta \leq \mho_{(k)}\,\Theta \leq \Theta = \mho^{(k)}\,\Theta \leq \mho^{(k-1)}\,\Theta \leq \ldots \leq \mho^{(1)}\,\Theta$$

Definition 7.9. *Let $\Theta \in$ NExt* ***S4****. Let k be such that Θ is of fatness k but not of fatness $k-1$. The* ***trimming space*** *of Θ, $T(\Theta)$, is the matching space of Θ and $\Omega_{(k+1)}\,\Theta$.*

FIGURE 5. The trimming space of $\mathfrak{F}\mathfrak{k}_3$



If $\mathfrak{F}$ is the generating frame of Θ, we shall also speak of $T(\Theta)$ as the trimming space of $\mathfrak{F}$. The trimming space of $\mathfrak{F}$ is the set of all irreducible logics $\mathsf{Th}\,\mathfrak{G}$ where $\mathfrak{G}$ is rooted such that (1) $\mathfrak{G} \leq \mathfrak{F}$, (2) $\mathfrak{G}$ is of fatness $\leq k+1$ and (3) $\mho^{(1)}\mathfrak{G} \cong \mho^{(1)}\mathfrak{F}$. The following is clear.

Lemma 7.10. *Let $\mathfrak{F}$ be a finite rooted slender **S4**–frame. Then the trimming space is l–constructible in NExt **S4**.*

Notice also that the property of slenderness is l–definable. The trimming space will be of cardinal importance in recovering the structure of $\mathfrak{F}$. Notice first of all that the trimming space has a largest element, $\mathsf{Th}\,\mathfrak{F}$, and a lowest element, $\mathsf{Th}\,\Omega_{(k+1)}\mathfrak{F}$. The dimension is defined in such a way that the highest element has lowest dimension. This is due to the geometrical intuition that underlies the trimming space. We analyse first the trimming space of slender frames. If $\mathfrak{F}$ is slender, the trimming space consists of all logics of frames $\mathfrak{G}$ obtained from $\mathfrak{F}$ by increasing any number of clusters by one point. We will however restrict our attention to logics of the form $\mathsf{Th}\,\boxplus_C\,\mathfrak{F}$ and of the form $\mathsf{Th}\,\boxplus_C \boxplus_D\,\mathfrak{F}$, where C and D are distinct clusters. It is easily seen that the first of them have codimension 1 and the second has codimension 2.

Definition 7.11. *Let Θ be the logic of a finite rooted slender **S4**–frame. A **point** in the trimming space is an element of codimension (!) 1; a **line** is an element of codimension 2 which is below the frame $\langle 1,2,2\rangle$ or $\langle 2,2\rangle$. The **trimming plane** of $\mathfrak{F}$ is the triple $\langle P(\mathfrak{F}), L(\mathfrak{F}), I\rangle$, where $P(\mathfrak{F})$ is the set of points, $L(\mathfrak{F})$ the set of lines, and $I \subseteq P(\mathfrak{F}) \times L(\mathfrak{F})$ is defined by $\mathsf{P}\ I\ \mathsf{L}$ iff $\mathsf{P} \geq \mathsf{L}$, for all $\mathsf{P} \in P(\mathfrak{F})$ and $\mathsf{L} \in L(\mathfrak{F})$.*

The reader may check that the trimming plane is also l–constructible. An example of a trimming space is shown in Figure 5. Let us look at slender frames first. An element of dimension 1 is the logic of a frame $\mathfrak{G}$ which has one more point than $\mathfrak{F}$ and the same skeleton. Hence, $\mathfrak{G}$ contains somewhere a proper cluster. It might seem that there are as many points in the trimming space as there are points in $\mathfrak{F}$, but this is not true. For if x and y are in the same orbit of the automorphism group, then the same logic arises if we blow up x to a proper cluster, or if we blow up y instead.

Lemma 7.12. *Let $\mathfrak{F}$ be slender **S4**–frames and $C = \{x\}$, $D = \{y\}$ be clusters. Then $\boxplus_C\,\mathfrak{F}$ and $\boxplus_D\,\mathfrak{F}$ are isomorphic iff there exists an automorphism of $\mathfrak{F}$ mapping C to D iff $y \in [x]$.*

Proof. Suppose there is an isomorphism $\pi : \boxplus_C\,\mathfrak{F} \to \boxplus_D\,\mathfrak{F}$. Let the cluster of x in $\boxplus_C\,\mathfrak{F}$ consist of the points x and x', and the cluster of y in $\boxplus_D\,\mathfrak{F}$ of the points y and y'. It is clear that $\pi[\{x,x'\}] = \{y,y'\}$. Hence two cases arise. Case (1). $\pi(x) = y$. Then $\pi \restriction \mathfrak{F}$ is an automorphism of $\mathfrak{F}$ mapping x to y. Case (2). $\pi(x) = y'$. Then define π' by $\pi'(x) := y$, $\pi'(x') := y'$ and $\pi'(z) := z$ for all other points. Then π' is an isomorphism from $\boxplus_C\,\mathfrak{F}$ to $\boxplus_D\,\mathfrak{F}$. (Namely, it is the composition of π with the automorphism of $\boxplus_C\,\mathfrak{F}$ which exchanges x and x'.) Now we are in Case (1). Namely, $\pi' \restriction \mathfrak{F}$ is an automorphism of $\mathfrak{F}$ mapping x to y. ∎

So, the trimming space has as many points as there are orbits in $\mathfrak{F}$. Hence, only if $\mathfrak{F}$ is rigid the set of points of the trimming space has the same cardinality as $\mathfrak{F}$, and the trimming map is injective.

Lemma 7.13. *Let $\mathfrak{F}$ be a finite rooted slender **S4**–frame. The trimming map is injective iff $\mathfrak{F}$ is rigid.*

In case $\mathfrak{F}$ is rigid, the structure of $\mathfrak{F}$ is recoverable from the trimming space. In general, only the structure of $\partial\,\mathfrak{F}$ can be determined in this way. For now look at the elements of codimension 2. These have exactly two covers in the trimming space, which are points. So, a line is in fact some two element subset of $P(\mathfrak{F})$. (Again, this will not hold in general.) Not any pair of points defines a line. Namely, two points are incident on a line exactly when the line lies below the linear frames $\langle 1, 2, 2\rangle$ or $\langle 2, 2\rangle$. However, this means exactly that the improper clusters are related via $\lhd$. This follows from the next theorem.

Lemma 7.14. *Let $\mathfrak{F}$ be a finite **S4**–frame. Then $\mathfrak{F}$ contains two different clusters C and D of fatness at least k with $C \lhd D$ iff $\mathfrak{F} \leq \langle 1, k, k\rangle$ or $\mathfrak{F} \leq \langle k, k\rangle$.*

Proof. Clearly, if $\mathfrak{F} \leq \langle 1, k, k\rangle$ or if $\mathfrak{F} \leq \langle k, k\rangle$, $\mathfrak{F}$ contains two clusters C and D of fatness $\geq k$ such that $C \lhd D$ and $C \neq D$. So, only the direction from left to right still needs a proof. Suppose C and D are clusters of $\mathfrak{F}$ of fatness at least k, $C \neq D$, and that $C \lhd D$. Let $\mathfrak{G}$ be the subframe generated by C. Without loss of generality we may assume that $\sharp C = \sharp D = k$. Case (1). D is not final. Take all clusters which cannot see D and map them to a single point. This is a p–morphism onto a frame of the form $C \oslash \mathfrak{K} \oslash D \oslash \circ$. Now, collapse $\mathfrak{K}$ into D. This yields the frame $C \oslash D \oslash \circ$, which is isomorphic to $\langle 1, k, k\rangle$. Case (2). D is final. Take all clusters different from C and collapse them into D. This is a p–morphism onto $\langle k, k\rangle$. ☐

Let P be a point in the trimming plane of $\mathfrak{F}$. Let us agree to write $dp(\mathsf{P}) = d$ if the (unique) proper cluster of the generating frame of P has depth d. This map is l–constructible in NExt **S4**. Hence, P_1 and P_2 are incident on a line iff the corresponding points of $\mathfrak{F}$ are connected via $\lhd$. Now, since $\mathfrak{F}$ is slender and rigid, we may identify points of the trimming plane with elements (= clusters) of $\mathfrak{F}$. This bijection is in fact the trimming map. So, let $\mathsf{P}_1 = \mathsf{Th}\boxplus_C \mathfrak{F}$ and $\mathsf{P}_2 = \mathsf{Th}\boxplus_D \mathfrak{F}$. Then $C \lhd D$ with $C \neq D$ iff (0) $\mathsf{P}_1 \neq \mathsf{P}_2$ (by rigidity), (1) P_1 and P_2 are incident on a line and (2) the depth of C is larger than the depth of D (by the previous lemma). The following is now proved.

Lemma 7.15. *Let $\mathfrak{F}$ be a finite, rooted, slender and rigid **S4**–frame, and let $\langle P(\mathfrak{F}), L(\mathfrak{F}), I\rangle$ be the trimming plane of $\mathfrak{F}$. Let $\mathsf{P}_1, \mathsf{P}_2 \in P(\mathfrak{F})$. Put $\mathsf{P}_1 \blacktriangleleft \mathsf{P}_2$ iff either $\mathsf{P}_1 = \mathsf{P}_2$ or: (a) P_1 and P_2 are on a line and (b) $dp(\mathsf{P}_1) > dp(\mathsf{P}_2)$. Then the trimming map is an isomorphism from $\mathfrak{F}$ onto $\langle P(\mathfrak{F}), \blacktriangleleft\rangle$.*

Corollary 7.16. *Let $\mathfrak{F}$ be a finite, slender and rigid **S4**–frame. Let α be an automorphism of NExt **S4**. Then α fixes $\mathsf{Th}\,\mathfrak{F}$.*

Proof. $\alpha(\mathfrak{F})$ is of fatness 1. Moreover, the cardinality of the points of the trimming space and the cardinality of $\mathfrak{F}$ are invariant. Hence, the trimming space of $\alpha(\mathfrak{F})$ has as many points as the trimming space of $\mathfrak{F}$, and $\alpha(\mathfrak{F})$ has as many points as $\mathfrak{F}$. It follows that $\alpha(\mathfrak{F})$ is rigid. Now, $\alpha(\mathfrak{F})$ is recoverable from the trimming space using comparison with frames which are invariant under automorphisms of NExt **S4**. It follows that $\alpha(\mathfrak{F}) \cong \mathfrak{F}$. ☐

So, we have shown that slender and rigid frames must be fixed. To extend this result to other frames, we observe that given $\mathfrak{F}$ there are frames below $\mathfrak{F}$ with identical skeleton which are rigid in a certain sense. Namely, we blow up the clusters in such a way that they end up having pairwise different cardinality. The resulting frame is rigid on the clusters: we can only permute the points of a cluster, but we cannot permute the clusters. This will help us to get a grip on the structure of the skeleton of $\mathfrak{F}$.

Definition 7.17. *Let $\mathfrak{F}$ be a **S4**–frame. Call $\mathfrak{F}$ n–**spread** if for all clusters C and D such that $D \neq C$, $|\sharp C - \sharp D| \geq n$.*

Clearly, since the weight functions are l–constructible in NExt **S4**, so are the properties of being n–spread. If a frame is $n+1$–spread it is also n–spread. Any frame is 0–spread. Any frame $\mathfrak{F}$ is a p–morphic image of some n–spread frame $\mathfrak{G}$ with identical skeleton for any given n. We can use this to show the following.

Lemma 7.18. *Suppose that $\mathfrak{F}$ is a finite and slender **S4**–frame. Then there exists some $\mathfrak{G}$ such that $\partial\,\mathfrak{G} \cong \mathfrak{F}$.*

Proof. Choose some $\mathfrak{G}$ which is 1–spread such that $\mho^{(1)}\,\mathfrak{G} \cong \mathfrak{F}$. An automorphism of $\mathfrak{G}$ may not map an element of some cluster C onto some element of some other cluster D. However, if x and y are elements of the same cluster, there exists an automorphism α of $\mathfrak{G}$ such that $\alpha(x) = y$. Hence $\partial\,\mathfrak{G} \cong \mho^{(1)}\,\mathfrak{G} \cong \mathfrak{F}$. □

Actually, for the last lemma finiteness is not needed.

Lemma 7.19. *Let $\mathfrak{F}$ be an **S4**–frame which is n–spread for some $n > 0$. Then there exists an isomorphism from $\boxplus_C\,\mathfrak{F}$ onto $\boxplus_D\,\mathfrak{F}$ iff $C = D$.*

Proof. Suppose that $\pi : \boxplus_C\,\mathfrak{F} \to \boxplus_D\,\mathfrak{F}$ is an isomorphism. We claim $\sharp C = \sharp D$. For assume not. Let $\sharp C < \sharp D$. Then $\mathfrak{F}$ contains a cluster E of same cardinality as D. So, $\boxplus_D\,\mathfrak{F}$ contains two clusters of cardinality $\sharp D$. But $\boxplus_C\,\mathfrak{F}$ contains only one such cluster. Contradiction. Hence, $\sharp C = \sharp D$, and since $\mathfrak{F}$ is n–spread with $n > 0$, $C = D$. The converse is straightforward. □

So, n–spread frames, where $n > 0$, are ideal targets for our investigation. Even though $n = 1$ would be enough for the previous theorem, we will concentrate on frames with $n \geq 2$. The reason is that if a frame is at least 2–spread then we can l–define the function $\mathsf{Th}\ \boxplus_C\ \mathfrak{F} \to \mathsf{Th}\ \boxminus_C\ \mathfrak{F}$.

Lemma 7.20. *Let $\mathfrak{F}$ be a 2–spread **S4**–frame and C and D clusters of $\mathfrak{F}$. If there is a number k such that $wt(\boxplus_C\,\mathfrak{F}) -_m \{k+1\} = wt(\boxminus_D\,\mathfrak{F}) -_m \{k-1\}$, then $C = D$.*

Proof. The weight of $\boxplus_C\,\mathfrak{F}$ is $wt(\mathfrak{F})$, where $\sharp C$ is replaced by $\sharp C + 1$. The weight of $\boxminus_D\,\mathfrak{F}$ is $wt(\mathfrak{F})$, where $\sharp D$ is replaced by $\sharp D - 1$. Now, let $P := wt(\boxplus_C\,\mathfrak{F})$ and $M := wt(\boxminus_D\,\mathfrak{F})$. P and M differ by only one element iff $C = D$. Otherwise, they differ by two elements. (The case $\sharp D = 1$ needs special attention, but causes no difficulty here.) □

The following is an immediate consequence of the preceding theorem.

Lemma 7.21. *Let $\mathfrak{F}$ be a rooted, 2–spread **S4**–frame. Then the map $\mathsf{Th}\ \boxplus_C\ \mathfrak{F} \to \mathsf{Th}\ \boxminus_C\ \mathfrak{F}$ is l–constructible in NExt **S4**.*

Definition 7.22. *Let $\mathfrak{F}$ be a frame of fatness k. A **point** of the trimming space is a maximal element with a cluster of size $k+1$. The **fatness** of the point P is defined by $f(P) := k+1-codim\,\mathsf{P}$. A **line** is a maximal element L with two clusters of size $k+1$ such that $\mathfrak{G} \leq \langle 1, k+1, k+1\rangle$ or $\mathsf{L} \leq \langle k+1, k+1\rangle$. The set of points is denoted by $P(\mathfrak{F})$, the set of lines by $L(\mathfrak{F})$. The **trimming plane** of $\mathfrak{F}$ is $\langle P(\mathfrak{F}), L(\mathfrak{F}), I\rangle$ where $\mathsf{P}\ I\ \mathsf{L}$ iff $\mathsf{L} \leq \mathsf{P}$.*

This definition generalizes the Definition 7.11. Again, it is clear that the trimming plane is l–constructible.

Lemma 7.23. *Let $\mathfrak{F}$ be a rooted, finite 1–spread **S4**–frame and $\Lambda \in T(\mathfrak{F})$. Λ is of the form $\mathsf{Th}\ \boxplus_C^k\ \mathfrak{F}$ for some k iff there is a point $\mathsf{P} \leq \Lambda$.*

Proof. We show that Λ is a point iff $\Lambda = \mathsf{Th}\ \boxplus_C^d\ \mathfrak{F}$, where $d := codim\,\mathsf{P}$. Suppose that $\sharp C = f$. Put $d := k+1-f$. Then $\boxplus_C^d\,\mathfrak{F}$ contains exactly one cluster of size $k+1$, and it is maximal in $T(\mathfrak{F})$ with this property. Put $\mathsf{P} := \mathsf{Th}\ \boxplus_x^d\ \mathfrak{F}$. Then any extension of P is of the form $\boxplus_C^m\,\mathfrak{F}$, $m \leq d$. Furthermore, $d = codim\,\mathsf{P}$ and $f = f(\mathsf{P})$. This shows one direction. For the other direction, assume that P is maximal with the property of containing a cluster of size $k+1$. P can be obtained by a series of operations $\boxplus_C$. Suppose that the cluster of P of size $k+1$ is C. Then $\boxplus^d_C\,\mathfrak{F} \geq \mathsf{P}$, where $d := \mathit{codim}\,\mathsf{P}$. Clearly, the frame $\boxplus^d_C\,\mathfrak{F}$ also contains a cluster of size $k+1$, hence it is isomorphic to the frame of P, by maximality of P. □

Proposition 7.24. *Let $\mathfrak{F}$ be a 1–spread rooted **S4**–frame of fatness k and let $\langle P(\mathfrak{F}), L(\mathfrak{F}), I\rangle$ be the trimming plane of $\mathfrak{F}$. Then put $K := \{\langle \mathsf{P}, i\rangle : \mathsf{P} \in P, i < f(\mathsf{P})\}$, and let $\langle \mathsf{P}, i\rangle \blacktriangleleft \langle \mathsf{Q}, j\rangle$ iff $\mathsf{P} = \mathsf{Q}$ or P and Q are on a line and the depth of the (unique) cluster of fatness $k+1$ of P is greater than the depth of the (unique) cluster of fatness $k+1$ of Q. Then $\langle K, \blacktriangleleft\rangle$ is isomorphic to $\mathfrak{F}$.*

Proof. Before we prove the theorem, let us note that $\langle K, \blacktriangleleft\rangle$ is constructible from the lattice. Namely, let P and Q be given. To know whether $\mathsf{P} \blacktriangleleft \mathsf{Q}$ we not only have to determine whether they are on a line, but also whether the cluster of depth $k+1$ in the generating frame of P has depth greater than the depth of the cluster of fatness $k+1$ occurring in the generating frame of Q. It follows from Lemma 7.3, that we can determine at which depth the cluster of size of $k+1$ in a point occurs. Furthermore, given P there is a unique cluster C such that $\mathsf{P} \leq \boxplus_C\,\mathfrak{F}$. Otherwise P would not be a maximal frame containing exactly one cluster of size $k+1$. Hence we have a bijection between points and cocovers, and the number $f(\mathsf{P})$ is unique. Therefore, $\langle K, \blacktriangleleft\rangle$ can be constructed (and is unique). Now define the following map. For each cluster C, let $\gamma_i : \sharp C \to C$ be a bijection. Furthermore, let $\zeta : C \mapsto P(\mathfrak{F})$ map each cluster C to the point $\mathsf{P} \leq \boxplus_C\,\mathfrak{F}$. Then the map $\beta : x \mapsto \langle \zeta(C), \gamma_i^{-1}(x)\rangle$ is well–defined and a bijection. From Lemma 7.14 we deduce that $\blacktriangleleft = \beta[\lhd]$. This concludes the proof. □

So we have managed to reconstruct $\mathfrak{F}$ from the trimming space of its logic, however on condition that $\mathfrak{F}$ is 1–spread in addition to being rooted. We finally show that we can reconstruct $\mathfrak{F}$ even when it is not 1–spread. Clearly, we can concentrate on irreducible logics. Suppose that $\Lambda = \mathsf{Th}\,\mathfrak{F}$ is given. We look for a logic $\Theta = \mathsf{Th}\,\mathfrak{G}$ where $\mathfrak{G}$ is 2–spread, is below Λ and has the same skeleton as $\mathfrak{F}$, and has no improper clusters. It is not hard to see that Θ can be constructed using only the structure of the lattice. Namely, we know the fatness and cardinality of $\mathfrak{F}$, hence we can give an upper bound on the order codimension of $\mathfrak{G}$. Finally, we can decide, given Θ, whether $\mathfrak{G}$ has the same skeleton as $\mathfrak{G}$, whether $\mathfrak{G} \leq \mathfrak{F}$, and whether $\mathfrak{G}$ is 2–spread (simply look at the weight). From Θ the structure of $\mathfrak{G}$ is reconstructible. Moreover, we can determine the skeleton of $\mathfrak{F}$. What is still left to determine is the cardinality of the clusters of $\mathfrak{F}$.

We proceed as follows. The skeleton of $\mathfrak{F}$ will be $\langle \mathsf{P}(\mathfrak{G}), \blacktriangleleft\rangle$. We have a bijection from $\mathsf{P}(\mathfrak{G})$ to the set of cocovers in $T(\mathfrak{G})$, which are exactly the frames of the form $\boxplus_C\,\mathfrak{G}$. Using Lemma 7.20 we construct a bijection from the set of cocovers of $\mathfrak{G}$ in $T(\mathfrak{G})$ onto the set of covers of $\mathfrak{G}$ in the matching space $M(\mathfrak{G}, \mathfrak{F})$.

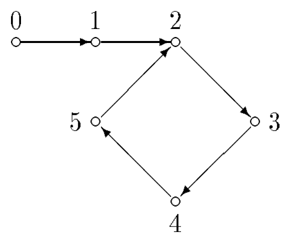
Definition 7.25. *Let $\mathfrak{F}$ be a finite rooted **S4**–frame and $\mathfrak{G} \leq \mathfrak{F}$ a frame with isomorphic skeleton. The **tower** over C, C a cluster of $\mathfrak{G}$, is the set T_C of elements the matching space $M(\mathfrak{F}, \mathfrak{G})$ which are above $\mathsf{Th}\ \boxminus_C\ \mathfrak{G}$ but not above any other atom. The cardinality of T_C is called the **height** of T_C.*

Lemma 7.26. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite **S4**–frames with identical skeleton, $\mathfrak{G} \leq \mathfrak{F}$, and $\mathfrak{G}$ 1–spread. Let $\langle C_i : i < m\rangle$ be an enumeration of the clusters of $\mathfrak{G}$. Further, let h_i be the height of the tower T_{C_i} in the matching space of $\mathfrak{F}$ and $\mathfrak{G}$. Then*

$$\mathfrak{F} \cong \boxminus^{h_0}_{C_0}\ \boxminus^{h_1}_{C_1} \ldots \boxminus^{h_{m-1}}_{C_{m-1}}\ \mathfrak{G}$$

Proof. Let $\Lambda \in T_C$, C a cluster of $\mathfrak{G}$. Then $\Lambda = \mathsf{Th}\,\mathfrak{P}$ for some frame $\mathfrak{P}$ such that $\boxminus_C\,\mathfrak{G} \leq \mathfrak{P}$. We know that $\mathfrak{P}$ is obtained from $\mathfrak{G}$ by a composition of inverse trimming maps. Now, it is clear that this composition is of the form $\boxminus^k_C$ for some k, otherwise $\Lambda \geq \mathsf{Th}\ \boxminus_D\ \mathfrak{G}$ for some $C \neq D$. Hence, $\mathfrak{P} \cong \boxminus^k_C\,\mathfrak{G}$ for some k. Let T_C have height h. Then the maximal element of T_C is the logic $\mathsf{Th}\ \boxminus^h_C\ \mathfrak{G}$. Hence, from the cluster C we must take exactly h elements. Now, $\mathfrak{F}$ is the least upper bound of these logics, and it is not hard to check that this least upper bound is the logic of the frame that is obtained from $\mathfrak{G}$ by removing from each cluster C exactly $\sharp T_C$ elements. □

FIGURE 6. The frame $\mathfrak{Loop}_{2,4}$



Theorem 7.27. *There exists an elementary function $f : \omega \longrightarrow \omega$ such that the following holds. For every logic $\Lambda \in NExt\,\boldsymbol{S4}$ of codimension n the generating frame of Λ is reconstructible up to isomorphism from the structure of the poset of logics of order codimension at most $f(n)$.*

In fact, let $\mathfrak{F}$ have n elements. Let its weight be $\{w_i : i < p\}$, where $w_i \leq w_j$ if $i < j$. Then $p \leq n$. Then the cluster sequence $\{w_i + 2i : i < p\}$ is 2–spread. $w_{p-1} + 2(p-1) < 3n$. Therefore there is a frame of fatness at most $3n$ which is 2–spread, below $\mathfrak{F}$ and has the same skeleton as $\mathfrak{F}$. Hence, the trimming space of $\mathfrak{G}$ consists of the frames of fatness $\leq 3n+1$ and skeleton size $\leq n$. These logics are of order codimension at most $3n^2 + n$. So, $f(n) := 3n^2 + n$ is a good choice.

We conclude with a series of corollaries.

Theorem 7.28. $\mathfrak{Aut}(NExt\,\boldsymbol{S4}^{\delta}) \cong 1$.

This means that the lattice of **S4**–logics of finite codimension is rigid.

Theorem 7.29. *Let α be an automorphism of $NExt\,\boldsymbol{S4}$. Then $Fix(\alpha)$ contains all elements of finite codimension.*

Corollary 7.30. *Let α be an automorphism of $NExt\,\boldsymbol{S4}$ and let Λ have the finite model property. Then $\Lambda \in Fix(\alpha)$.*

Denote by Λ_o the smallest logic having the same finite models as Λ, and by Λ^o the largest such logic. Λ^o is well–defined, being the intersection of all $\mathsf{Th}\,\mathfrak{F}$ where $\mathfrak{F}$ is rooted, finite and $\mathfrak{F} \models \Lambda$. Λ_o is well–defined. For

$$\Lambda_o = \bigsqcup \langle \mathbf{S4}/\mathfrak{F} : \mathfrak{F} \not\models \Lambda, \mathfrak{F}\ \textit{finite and rooted} \rangle$$

We call $[\Lambda_o, \Lambda^o]$ the **prime spectrum** of Λ. (This terminology is due to the fact that the logics of finite rooted frames are the prime elements of NExt **S4**.)

Theorem 7.31. *Every prime spectrum of $NExt\,\boldsymbol{S4}$ is fixed under an automorphism of $NExt\,\boldsymbol{S4}$. In particular, the maximal and the minimal element of a given spectrum are fixed.*

Of course, if α fixes all spectra pointwise, α is the identity. So, as far as the results go, we can only show that the spectra are fixed as sets, not necessarily pointwise.

8. AUTOMORPHISMS OF NEXT $\mathbf{K.alt_1}$

We will show in this section that the automorphism groups can be rather large. A particular case is $\mathfrak{Aut}\,\text{NExt}\,\mathbf{K.alt_1}$. Recall from [10] that all extensions have the finite model property, and that the rooted frames are of the following types. (a) The infinite chain $\mathfrak{Ich}_\infty$, (b) the finite chains $\mathfrak{Ich}_k$, $k \in \omega$, (c) the loops $\mathfrak{Loop}_{p,q}$. The infinite chain is the frame $\mathfrak{Ich}_\infty = \langle \omega, \lhd \rangle$ where $i \lhd j$ iff $i + 1 = j$. The finite chain $\mathfrak{Ich}_k$, is the initial segment of $\mathfrak{Ich}_\infty$ of length k. The loops $\mathfrak{Loop}_{p,q}$ are based on the set of numbers $< p+q$, and we have $i \lhd j$ iff $i+1 = j$ or $j = p$ and $i = p+q-1$. Hence, the loops contain a cycle of length q, and an initial segment of length p. Figure 6 shows the frame $\mathfrak{Loop}_{2,4}$. Let U be the set of all finite chains and loops. Put $\mathfrak{F} \leq \mathfrak{G}$ iff $\mathsf{Th}\,\mathfrak{F} \subseteq \mathsf{Th}\,\mathfrak{G}$. It turns out that chains and loops are incomparable, that $\mathfrak{Ich}_k \leq \mathfrak{Ich}_m$ iff $k \geq m$, and that $\mathfrak{Loop}_{p,q} \leq \mathfrak{Loop}_{r,s}$

iff $r \leq p$ and $s \mid q$ (s divides q). This defines the partial order of finite frames. Logics are identified by closed subsets of U. We will determine these sets as we proceed. First, notice that the chains are exactly the splitting frames of $\mathbf{K.alt}_1$. Their companions are the so–called Chellas–Hughes logics

$$\mathbf{CH}_k := \mathbf{K.alt}_1/\mathfrak{Ch}_k$$

(This name is taken from [10].) In particular, $\mathbf{CH}_1 = \mathbf{K.alt}_1.\mathbf{D}$.

Lemma 8.1. *Every automorphism of NExt* $\boldsymbol{K.alt}_1$ *fixes the Chellas–Hughes logics and the logics of chains.*

A logic extending $\mathbf{K.alt}_1$ is either an extension of $\mathbf{K.alt}_1.\mathbf{D}$ or it is the intersection of an extension of $\mathbf{K.alt}_1.\mathbf{D}$ and the logic of a chain. Since the chains are fixed, we obtain the following result.

Theorem 8.2. $\mathfrak{Aut}(\mathit{NExt}\,\boldsymbol{K.alt}_1.\boldsymbol{D}) \cong \mathfrak{Aut}(\mathit{NExt}\,\boldsymbol{K.alt}_1)$. *Moreover, the following holds.*

1. *Every automorphism of NExt* $\boldsymbol{K.alt}_1$ *induces an automorphism of NExt* $\boldsymbol{K.alt}_1.\boldsymbol{D}$.
2. *Every automorphism of NExt* $\boldsymbol{K.alt}_1.\boldsymbol{D}$ *can be uniquely extended to an automorphism of NExt* $\boldsymbol{K.alt}_1$.

A logic properly containing NExt $\mathbf{K.alt}_1.\mathbf{D}$ is tabular. Hence, the closed sets of U are as follows. $M \subseteq U$ is closed iff it is upward closed and (a) M is a finite set of loops and a finite set of chains, or (b) M contains all loops and a finite set of chains, or (c) M contains all loops and all chains.

By the previous theorem, we need to study only the group of automorphisms of $\mathbf{K.alt}_1.\mathbf{D}$. Therefore, we may restrict our attention to the closed sets of loops. Now denote by L the set of loops. It is easy to see that there is a bijective correspondence between automorphisms of $\langle L, \leq\rangle$ and automorphisms of NExt $\mathbf{K.alt}_1.\mathbf{D}$. For all we need to see is that an order automorphism is also continuous. But this is clear: the closed sets are the finite subsets L and L itself. These sets are invariant under any order automorphism. Hence, even though the topology of the spectrum is not the Alexandrov–topology, the automorphisms of the locale are those of the underlying poset.

We are left with the problem of determining the automorphisms of $\langle L, \leq\rangle$. We have $\mathfrak{Loop}_{p,q} \leq \mathfrak{Loop}_{r,s}$ iff $r \leq p$ and $s \mid q$. There is exactly one element of codimension 1 (in NExt $\mathbf{K.alt}_1.\mathbf{D}$), namely $\mathfrak{Loop}_{0,1}$. The elements of codimension 2 are $\mathfrak{Loop}_{1,1}$ and $\mathfrak{Loop}_{0,q}$, q a prime number. We call Δ the set of these elements.

Lemma 8.3. *Any permutation of* Δ *can be uniquely extended to an automorphism of* $\langle L, \leq\rangle$.

Proof. Let P be the set of elements of L with covering number 1. We claim that

$$P = \{\mathfrak{Loop}_{n,1} : n \in \omega, n > 0\} \cup \{\mathfrak{Loop}_{0,q} : q \text{ a prime power}\}$$

Let $\mathfrak{Loop}_{p,q}$ have only one (order) cover. Assume $p > 0$. Then $\mathfrak{Loop}_{p-1,q}$ is a cover of $\mathfrak{Loop}_{p,q}$. Let r be a maximal divisor of q. Then $\mathfrak{Loop}_{p,r}$ is another cover of $\mathfrak{Loop}_{p,q}$. Hence $q = 1$. Now assume that $p = 0$. Suppose that q is not the power of a prime. Then $q = ab$ for some relatively prime a and b. Then

$$\mathfrak{Loop}_{p,q} = glb\,\{\mathfrak{Loop}_{p,a}, \mathfrak{Loop}_{p,b}\}$$

Therefore, the element $\mathfrak{Loop}_{p,q}$ has more than one cover. Hence, q is a prime power. Now, for other direction assume that $p > 0$ and $q = 1$. Then $\mathfrak{Loop}_{p-1,1}$ in the only cover of $\mathfrak{Loop}_{p,q}$. Assume next that $q = q_*^k$, $k > 0$ and q_* a prime number. Then $\mathfrak{Loop}_{p,q_*^{k-1}}$ is the unique cover of $\mathfrak{Loop}_{p,q}$. So, P is the set of elements with exactly one cover.

For each element of P, $\mathfrak{F}$, there exists a unique $\mathfrak{G} \in \Delta$ such that $\mathfrak{G} \geq \mathfrak{F}$. The order on P is a disjoint union of orders of the form $\omega^\perp$, each maximal member corresponding to an element of Δ. Hence any permutation of Δ extends to a unique automorphism of $\langle P, \leq\rangle$.

Now observe that

$$\mathfrak{F} = glb\,\{\mathfrak{G} : \mathfrak{G} \in P, \mathfrak{G} \geq \mathfrak{F}\}$$

For let $\mathfrak{F} = \mathfrak{Loop}_{p,q}$. Since q can be decomposed into a product of prime powers with distinct base, the claim follows from the following facts: (a) $glb\,\{\mathfrak{Loop}_{0,q}, \mathfrak{Loop}_{p,1}\} = \mathfrak{Loop}_{p,q}$, (b) *glb* $\{\mathfrak{Loop}_{0,q_1}, \mathfrak{Loop}_{0,q_2}\} = \mathfrak{Loop}_{0,q_1 q_2}$ if q_1 and q_2 are relatively prime. Hence, each frame corresponds to an upward closed set in P. This correspondence is unique. Moreover, it turns out that $\langle L, \leq\rangle \cong \langle \wp^\star(P), \subseteq\rangle$, where $\wp^\star(P)$ is the set of finite, upward closed subsets of P. Therefore each automorphism of P gives rise to an automorphism of $\langle L, \leq\rangle$. $\Box$

The following is now immediate.

Theorem 8.4. $\mathfrak{Aut}\, \mathit{NExt}\, \boldsymbol{K.alt}_1.\boldsymbol{D} \cong \mathit{Sym}(\aleph_0)$.

From this theorem we obtain

Corollary 8.5. $\mathfrak{Aut}\, \mathit{NExt}\, \boldsymbol{K.alt}_1 \cong \mathit{Sym}(\aleph_0)$.

We note the following fact.

Theorem 8.6. *Let Λ be a consistent logic properly containing $\boldsymbol{D.alt}_1$. Then the l–indeterminacy of Λ with respect to $\mathit{NExt}\, \boldsymbol{D.alt}_1$ is $\aleph_0$.*

The proof is rather easy. All proper extensions are finitely axiomatizable, so the l–indeterminacy is $\leq \aleph_0$. The orbit of any coirreducible logic is infinite, as we have seen. Hence Λ is mapped onto infinitely many logics, since all proper extensions of $\mathbf{D.alt}_1$ are characterized by finitely many coirreducibles. With respect to $\mathbf{K.alt}_1$ the facts are a little bit more subtle.

Theorem 8.7. *Let Λ be a consistent logic properly containing $\boldsymbol{D.alt}_1$. Then the l–indeterminacy of Λ with respect to $\mathit{NExt}\, \boldsymbol{K.alt}_1$ is either 1 or $\aleph_0$. It is 1 exactly in the case where Λ is the logic of a chain or a Chellas–Hughes logic.*

The results of this section can be exploited to show that a great variety of groups are automorphism groups of some lattices of extensions. We start with the symmetric groups.

Lemma 8.8. *Let P be a set of prime numbers with cardinality n. Let $k \in \omega$ and let $\Pi(P, k)$ be the logic of the frames $\mathfrak{Loop}_{0,p^k}$, $p \in P$. Then $\mathfrak{Aut}\, \mathit{NExt}\, \Lambda \cong \mathit{Sym}(n)$.*

Theorem 8.9. *Let G be a finite product of finite symmetric groups. Then there exists a modal logic Λ such that $G \cong \mathfrak{Aut}(\mathit{NExt}\, \Lambda)$.*

Proof. Let $G \cong \prod_{i<n} \mathit{Sym}(m_i)$. Choose pairwise disjoint sets P_i of prime numbers such that $\sharp P_i = m_i$ for $i < n$. Then let $\Lambda := \bigsqcap_{i<n} \Pi(P_i, i)$. An automorphism of $\mathrm{NExt}\, \Lambda$ is uniquely defined by an automorphism of $\mathfrak{Spec}(\mathrm{NExt}\, \Lambda)$. It is easy to see that any automorphism of Λ fixes the logics $\Pi(P_i, i)$, and therefore is determined by its action on the lattice $\mathrm{NExt}\, \Pi(P_i, i)$. The rest immediately follows. $\Box$

This can be generalized. Recall that a **graph** is a pair $\langle E, K\rangle$, where E is a nonempty set, the set of **vertices** and K a set of two–element subsets of E, called **edges**.

Theorem 8.10. *Let G a finite product of automorphism groups of finite graphs. Then there exists a logic Λ such that $G \cong \mathfrak{Aut}\, \mathit{NExt}\, \Lambda$.*

First, let G be the automorphism group of a graph. For a proof, we may assume that E is a set of prime numbers. Then Λ is defined to be the intersection of $\mathsf{Th}\, \mathfrak{Loop}_{0,p}$, $p \in E$, and the logics $\mathsf{Th}\, \mathfrak{Loop}_{0,pq}$, $\{p, q\} \in K$. It is easily verified that there is an isomorphisms from $\mathfrak{Aut}\, \langle E, K\rangle$ onto $\mathfrak{Aut}\, \mathrm{NExt}\, \Lambda$. If G is a finite product of automorphism groups, observe that we may choose E a set of prime powers p^i for some fixed i instead. Now reason as above. Examples of groups covered by this theorem are the dihedral groups.

A somewhat more delicate example are groups arising as automorphism groups of finite t–designs. A (**simple**) t–**design** is a pair $\langle P, B\rangle$ where P is a nonempty set and $B \subseteq \wp(P)$ such that (1) all members of B have the same cardinality and (2) there is a number λ such that for each set $T \subseteq P$ of cardinality t there exist exactly λ elements of B containing T. (See [1].) If $\lambda = 1$ we speak of a **Steiner triple**. Any finite simple t–design is a 2–design as can easily be shown. Other examples of designs are the finite projective planes, which are 2–designs with $\lambda = 1$.

Proposition 8.11. *Let G be the automorphism group of a finite simple t–design. Then there is a modal logic Λ such that $G \cong \mathfrak{Aut}\, NExt\, \Lambda$.*

Proof. Let $G \cong \mathfrak{Aut}(\langle P, L\rangle)$, where $\langle P, L\rangle$ is a t–design. Let Q be a set of primes, and $\beta : P \to Q$ be a bijection. Now let

$$R := \{\beta[U] : U \subseteq T \in L\}$$

The set $W := \{\mathfrak{Loop}_{0,p} : p \in R\}$, is upwards closed. Finally, put

$$\Lambda := \textstyle\bigsqcap_{\mathfrak{V}\in W} \mathsf{Th}\,\mathfrak{V} = \bigsqcap_{p\in R} \mathsf{Th}\,\mathfrak{Loop}_{0,p}$$

g is an automorphism of NExt Λ iff it is an automorphism of $\langle W, \leq\rangle$ iff it is an automorphism of $\langle P, L\rangle$. This gives the claim. □

Groups covered by this theorem are $A\Gamma L_d(q)$, $P\Gamma L_d(q)$, and the Mathieu groups. (See [3].) The previous result can be extended to finite products of such groups, by observing first that we could have taken Q a set of powers of primes, as in the example with symmetric groups. No doubt these results can be improved even further.

9. Conclusion

We have established that the group of automorphisms of NExt **S4.3** is isomorphic to $\mathbb{Z}_2$ and that the group of automorphisms of NExt $\mathbf{K.alt_1}$ is isomorphic to $Sym(\aleph_0)$. Furthermore, every automorphism of NExt **S4** fixes all elements of finite codimension and hence all tabular logics and all logics with the finite model property. The greatest obstacle in improving these results is the fact that we have no good knowledge about the lattice of **S4**–logics. It might seem that if we are only interested in the automorphism group of this lattice we need not know its structure too well, but at present we see no way to determine the group of automorphisms independently from the structure of the lattice. It seems feasible to show that the lattice of elements of finite codimension of the lattice of **K4**–logics are fixed under every automorphism. To see that, one needs to establish first that every automorphism of NExt **K4** fixes **S4**, so that we know already that it must be the identity on the upper part of **S4**.

We end the paper with a series of conjectures, in order of increasing difficulty.

Conjecture 9.1. *The lattice of **S4**–logics of width n is rigid for every n.*

Conjecture 9.2. *The lattice of **K4**–logics of finite codimension is rigid.*

Conjecture 9.3. *The lattice of **K4**–logics is rigid.*

Conjecture 9.4. *The lattice of normal modal logics is rigid.*

The last conjecture is the most interesting one for many reasons. For if it is true then a normal modal logic is uniquely identified by its place in the lattice of normal modal logics.

A related question is whether the lattice of intermediate logics is rigid. Since this lattice is isomorphic to NExt **Grz**, we may ask whether our results on NExt **S4** extend to the lattice NExt **Grz**. However, only intrinsically l–definable properties do not depend on the lattice in which a logic is embedded. For example, in NExt **Grz** cardinality is l–definable, since it is intrinsically l–definable in NExt **S4**. Likewise the property of being a handle. However, many constructions have made heavy use of blowing up clusters, so are not intrinsic in the sense of the definition. Our preliminary results (only partly contained here) seem to support the

Conjecture 9.5. *The lattice of intermediate logics of finite codimension is rigid.*

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